

# On Parametrization of the Linear $GL(4, C)$ and Unitary $SU(4)$ Groups in Terms of Dirac Matrices<sup>\*</sup>

Victor M. RED'KOV, Andrei A. BOGUSH and Natalia G. TOKAREVSKAYA

*B.I. Stepanov Institute of Physics, National Academy of Sciences of Belarus, Minsk, Belarus*

E-mail: [redkov@dragon.bas-net.by](mailto:redkov@dragon.bas-net.by), [bogush@dragon.bas-net.by](mailto:bogush@dragon.bas-net.by), [tokarev@dragon.bas-net.by](mailto:tokarev@dragon.bas-net.by)

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**Abstract.** Parametrization of  $4 \times 4$ -matrices  $G$  of the complex linear group  $GL(4, C)$  in terms of four complex 4-vector parameters  $(k, m, n, l)$  is investigated. Additional restrictions separating some subgroups of  $GL(4, C)$  are given explicitly. In the given parametrization, the problem of inverting any  $4 \times 4$  matrix  $G$  is solved. Expression for determinant of any matrix  $G$  is found:  $\det G = F(k, m, n, l)$ . Unitarity conditions  $G^+ = G^{-1}$  have been formulated in the form of non-linear cubic algebraic equations including complex conjugation. Several simplest solutions of these unitarity equations have been found: three 2-parametric subgroups  $G_1, G_2, G_3$  – each of subgroups consists of two commuting Abelian unitary groups; 4-parametric unitary subgroup consisting of a product of a 3-parametric group isomorphic  $SU(2)$  and 1-parametric Abelian group. The Dirac basis of generators  $\Lambda_k$ , being of Gell-Mann type, substantially differs from the basis  $\lambda_i$  used in the literature on  $SU(4)$  group, formulas relating them are found – they permit to separate  $SU(3)$  subgroup in  $SU(4)$ . Special way to list 15 Dirac generators of  $GL(4, C)$  can be used  $\{\Lambda_k\} = \{\alpha_i \oplus \beta_j \oplus (\alpha_i V \beta_j = \mathbf{K} \oplus \mathbf{L} \oplus \mathbf{M})\}$ , which permit to factorize  $SU(4)$  transformations according to  $S = e^{i\vec{a}\vec{\alpha}} e^{i\vec{b}\vec{\beta}} e^{i\mathbf{k}\mathbf{K}} e^{i\mathbf{l}\mathbf{L}} e^{i\mathbf{m}\mathbf{M}}$ , where two first factors commute with each other and are isomorphic to  $SU(2)$  group, the three last ones are 3-parametric groups, each of them consisting of three Abelian commuting unitary subgroups. Besides, the structure of fifteen Dirac matrices  $\Lambda_k$  permits to separate twenty 3-parametric subgroups in  $SU(4)$  isomorphic to  $SU(2)$ ; those subgroups might be used as bigger elementary blocks in constructing of a general transformation  $SU(4)$ . It is shown how one can specify the present approach for the pseudounitary group  $SU(2, 2)$  and  $SU(3, 1)$ .

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## 1 Introduction

The unitary groups play an important role in numerous research areas: quantum theory of massless particles, cosmology models, quantum systems with dynamical symmetry, nano-scale physics, numerical calculations concerning entanglement and other quantum information parameters, high-energy particle theory – let us just specify these several points:

- $SU(2, 2)$  and conformal symmetry, massless particles [7, 19, 20, 39, 72];
- classical Yang–Mills equations and gauge fields [64];
- quantum computation and control, density matrices for entangled states [2, 31, 65];
- geometric phases and invariants for multi-level quantum systems [55];

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- high-temperature superconductivity and antiferromagnets [36, 52];
- composite structure of quarks and leptons [67, 68, 51];
- $SU(4)$  gauge models [73, 29];
- classification of hadrons and their interactions [30, 34, 28].

Because of so many applications in physics, various parametrizations for the group elements of unitary group  $SU(4)$  and related to it deserve special attention. Our efforts will be given to extending some classical technical approaches proving their effectiveness in simple cases of the linear and unitary groups  $SL(2, C)$  and  $SU(2)$ , so that we will work with objects known by every physicist, such as Pauli and Dirac matrices. This paper, written for physicists, is self-contained in that it does not require any previous knowledge of the subject nor any advanced mathematics.

Let us start with the known example of spinor covering for complex Lorentz group: consider the 8-parametric  $4 \times 4$  matrices in the quasi diagonal form [18, 32, 45]

$$G = \begin{vmatrix} k_0 + \mathbf{k}\vec{\sigma} & 0 \\ 0 & m_0 - \mathbf{m}\vec{\sigma} \end{vmatrix}.$$

The composition rules for parameters  $k = (k_0, \mathbf{k})$  and  $m = (m_0, \mathbf{m})$  are

$$\begin{aligned} k_0'' &= k_0' k_0 + \mathbf{k}' \mathbf{k}, & \mathbf{k}'' &= k_0' \mathbf{k} + \mathbf{k}' k_0 + i \mathbf{k}' \times \mathbf{k}, \\ m_0'' &= m_0' m_0 + \mathbf{m}' \mathbf{m}, & \mathbf{m}'' &= m_0' \mathbf{m} + \mathbf{m}' m_0 - i \mathbf{m}' \times \mathbf{m}. \end{aligned}$$

With two additional constraints on 8 quantities  $k_0^2 - \mathbf{k}^2 = +1$ ,  $m_0^2 - \mathbf{m}^2 = +1$ , we will arrive at a definite way to parameterize a double (spinor) covering for complex Lorentz group  $SO(4, C)$ . At this, the problem of inverting of the  $G$  matrices with unit determinant  $\det G$  is solved straightforwardly:  $G = G(k_0, \mathbf{k}, m_0, \mathbf{m})$ ,  $G^{-1} = G(k_0, -\mathbf{k}, m_0, -\mathbf{m})$ . Transition from covering 4-spinor transformations to 4-vector ones is performed through the known relationship  $G\gamma^a G^{-1} = \gamma^c L_c^a$  which determine  $2 \Rightarrow 1$  map from  $\pm G$  to  $L$ .

There exists a direct connection between the above 4-dimensional vector parametrization of the spinor group  $G(k_a, m_a)$  and the Fedorov's parametrization [32] of the group of complex orthogonal Lorentz transformations in terms of 3-dimensional vectors  $\mathbf{Q} = \mathbf{k}/k_0$ ,  $\mathbf{M} = \mathbf{m}/m_0$ , with the simple composition rules for vector parameters

$$\mathbf{Q}'' = \frac{\mathbf{Q} + \mathbf{Q}' + i \mathbf{Q}' \times \mathbf{Q}}{1 + \mathbf{Q}' \mathbf{Q}}, \quad \mathbf{M}'' = \frac{\mathbf{M} + \mathbf{M}' - i \mathbf{M}' \times \mathbf{M}}{1 + \mathbf{M}' \mathbf{M}}.$$

Evidently, the pair  $(\mathbf{Q}, \mathbf{M})$  provides us with possibility to parameterize correctly orthogonal matrices only. Instead, the  $(k_a, m_a)$  represent correct parameters for the spinor covering group. When we are interested only in local properties of the spinor representations, no substantial differences between orthogonal groups and their spinor coverings exist. However, in opposite cases global difference between orthogonal and spinor groups may be very substantial as well as correct parametrization of them.

Restrictions specifying the spinor coverings for orthogonal subgroups are well known [32]. In particular, restriction to real Lorentz group  $O(3, 1)$  is achieved by imposing one condition (including complex conjugation)  $(k, m) \Rightarrow (k, k^*)$ . The case of real orthogonal group  $O(4)$  is achieved by a formal change (transition to real parameters)  $(k_0, \mathbf{k}) \Rightarrow (k_0, i\mathbf{k})$ ,  $(m_0, \mathbf{m}) \Rightarrow (m_0, i\mathbf{m})$ , and the real orthogonal group  $O(2, 2)$  corresponds to transition to real parameters according to  $(k_0, k_1, k_2, k_3) \Rightarrow (k_0, k_1, k_2, ik_3)$ ,  $(m_0, m_1, m_2, m_3) \Rightarrow (m_0, m_1, m_2, im_3)$ .

To parameterize 4-spinor and 4-vector transformations of the complex Lorentz group one may use curvilinear coordinates. The simplest and widely used ones are Euler's complex angles

(see [32] and references in [18]). In general, on the basis of the analysis given by Olevskiy [58] about coordinates in the real Lobachevski space, one can propose 34 different complex coordinate systems appropriate to parameterize the complex Lorentz group and its double covering.

A particular, Euler angle parametrization is closely connected with cylindrical coordinates on the complex 3-sphere, one of 34 possible coordinates. Such complex cylindrical coordinates can be introduced by the following relations [18]:

$$\begin{aligned} k_0 &= \cos \rho \cos z, & k_3 &= i \cos \rho \sin z, & k_1 &= i \sin \rho \cos \phi, & k_2 &= i \sin \rho \sin \phi, \\ m_0 &= \cos R \cos Z, & m_3 &= i \cos R \sin Z, & m_1 &= i \sin R \Phi, & m_2 &= i \sin R \sin \Phi. \end{aligned}$$

Here 6 complex variables are independent,  $(\rho, z, \phi)$ ,  $(R, Z, \Phi)$ , additional restrictions are satisfied identically by definition. Instead of cylindrical coordinates in  $(\rho, z, \phi)$  and  $(R, Z, \Phi)$  one can introduce Euler's complex variables  $(\alpha, \beta, \gamma)$  and  $(A, B, \Gamma)$  through the simple linear formulas:

$$\alpha = \phi + z, \quad \beta = 2\rho, \quad \gamma = \phi - z, \quad A = \Phi + Z, \quad B = 2R, \quad \Gamma = \Phi - Z.$$

Euler's angles  $(\alpha, \beta, \gamma)$  and  $(A, B, \Gamma)$  are referred to  $k_a, m_a$ -parameters by the formulas (see in [32])

$$\begin{aligned} \cos \beta &= k_0^2 - k_3^2 + k_1^2 + k_2^2, & \sin \beta &= 2\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}, \\ \cos \alpha &= \frac{-ik_0k_1 + k_2k_3}{\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}}, & \sin \alpha &= \frac{-ik_0k_2 - k_1k_3}{\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}}, \\ \cos \gamma &= \frac{-ik_0k_1 - k_2k_3}{\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}}, & \sin \gamma &= \frac{-ik_0k_2 + k_1k_3}{\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}}, \\ \cos B &= m_0^2 - m_3^2 + m_1^2 + m_2^2, & \sin B &= 2\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}, \\ \cos A &= \frac{+im_0m_1 + m_2m_3}{\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}}, & \sin A &= \frac{+im_0m_2 - m_1m_3}{\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}}, \\ \cos \Gamma &= \frac{+im_0m_1 - m_2m_3}{\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}}, & \sin \Gamma &= \frac{+im_0m_2 + m_1m_3}{\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}}. \end{aligned}$$

Complex Euler's angles as parameters for complex Lorentz group  $SO(4, C)$  have a distinguished feature: 2-spinor constituents are factorized into three elementary Euler's transforms ( $\sigma^i$  stands for the known Pauli matrices):

$$\begin{aligned} B(k) &= e^{-i\sigma^3\alpha/2} e^{i\sigma^1\beta/2} e^{+i\sigma^3\gamma/2} \in SL(2, C), \\ B(\bar{m}) &= e^{-i\sigma^3\Gamma/2} e^{i\sigma^1B/2} e^{+i\sigma^3A/2} \in SL(2, C)'. \end{aligned}$$

The main question is how to extend possible parameterizations of small orthogonal group  $SO(4, C)$  and its double covering to bigger orthogonal and unitary groups<sup>1</sup>. To be concrete we are going to focus attention mainly on the group  $SU(4)$  and its counterparts  $SU(2, 2)$ ,  $SU(3, 1)$ .

There exist many publications on the subject, a great deal of facts are known – in the following we will be turning to them. A good classification of different approaches in parameterizing finite transformations of  $SU(4)$  was done in the recent paper by A. Gsponer [35]. Recalling it, we will try to cite publications in appropriate places though many of them should be placed in several different subclasses – it is natural because all approaches are closely connected to each other.

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<sup>1</sup>In this subject, especially concerned with generalized Euler angles, we have found out much from Murnaghan's book [56].

- **Canonical form** [53, 54, 56, 57, 60, 61, 75]. They use explicitly the full set of the Lie generators<sup>2</sup> so that the group element is expressed as the exponential of the linear combination

$$G = \exp_n[i(a_1\lambda_1 + \cdots + a_n\lambda_N)]$$

the infinite series of terms implied by the exp – symbol is usually very difficult to be summed in closed form – though there exists many interesting examples of those:

- **Non-canonical forms** [3, 4, 8, 9, 11, 35, 38, 40, 42, 57, 59, 61, 63, 74]. As a consequence of the Baker–Campbell–Hausdorff theorem [1, 26, 37] it is possible to break-down the canonical form into a product

$$G = \exp_{n^{(1)}} \times \cdots \times \exp_{n^{(k)}}, \quad n^{(1)} + \cdots + n^{(k)} = N.$$

with the hope that  $\exp_{n^{(i)}}$  could be summed in closed form and also that these factors have simple properties. This possibility for the groups  $SU(4)$  and  $SU(2, 2)$  will be discussed in more detail in sections below.

- **Product form** [12, 13, 14, 16, 27, 38, 56, 57]. An extreme non-canonical form is to factorize the general exponential into a product of  $n$  simplest 1-parametric exponentials

$$G = \exp[ia_1\lambda_1] \times \cdots \times \exp[ia_N\lambda_N].$$

- **Basic elements** (the main approach in the present treatment) [8, 9, 35, 40, 44, 45, 46, 47, 74]. This way is to expand the elements of the group (matrices or quaternions) into a sum over basis elements and to work with a linear decomposition of the matrices over basic ones:

$$\begin{aligned} G' &= x'_n \lambda_m, & G &= x_n \lambda_m, & \lambda_0 &= I, & k &\in \{0, 1, \dots, N\}, \\ G'' &= G'G, & x''_k \lambda_k &= x'_m \lambda_m x_n \lambda_n = x'_m x_n \lambda_m \lambda_n, \end{aligned} \quad (1.1)$$

as by definition the relationships  $\lambda_m \lambda_n = e_{mnk} \lambda_k$  must hold, the group multiplication rule for parameters  $x_k$  looks

$$x''_k = e_{mnk} x'_m x_n. \quad (1.2)$$

The main claim is that the all properties of any matrix group are straightforwardly determined by the bilinear function, the latter is described by structure constants  $e_{mnk}$  entering the multiplication rule  $\lambda_m \lambda_n = e_{mnk} \lambda_k$ .

- **Hamilton–Cayley form** [8, 9, 11, 12, 13, 14, 15, 16, 17, 33, 63]. It is possible to expand the elements of the group into a power series of linear combination of generators:

$$\lambda(a) = i(a_1\lambda_1 + \cdots + a_N\lambda_N),$$

because of Hamilton–Cayley theorem this series has three terms for  $SU(3)$  and four terms for  $SU(4)$ :

$$\begin{aligned} SU(3), \quad G(a) &= e_0(a)I + e_1(a)\lambda(a) + e_2(a)\lambda^2(a), \\ SU(4), \quad G(a) &= e_0(a)I + e_1(a)\lambda(a) + e_2(a)\lambda^2(a) + e_3(a)\lambda^3(a). \end{aligned}$$

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<sup>2</sup>In the paper we will designate generators in Dirac basis by  $\Lambda_i$  whereas another set of generators mainly used in the literature will be referred as  $\lambda_i$ .

- **Euler-angles representations** [10, 21, 22, 23, 24, 25, 35, 42, 56, 57, 69]. In Euler-angles representations only a sub-set  $\{\lambda_n\} \subset \{\lambda_N\}$  of the Lie generators are sufficient to produce the whole set (for  $SU(N)$  we need only  $2(N-1)$  generators). In that sense all other way to obtain the whole set of elements are not minimal.

In our opinion, we should search the most simplicity in mathematical sense while working with basic elements  $\lambda_k$  and the structure constants determining the group multiplication rule (1.1), (1.2).

The material of this paper is arranged as follows.

In Section 2 an arbitrary  $4 \times 4$  matrix  $G \in GL(4, C)$  is decomposed into sixteen Dirac matrices<sup>3</sup>

$$G = AI + iB\gamma^5 + iA_l\gamma^l + B_l\gamma^l\gamma^5 + F_{mn}\sigma_{mn} = \begin{vmatrix} k_0 + \mathbf{k}\vec{\sigma} & n_0 - \mathbf{n}\vec{\sigma} \\ -l_0 - \mathbf{l}\vec{\sigma} & m_0 - \mathbf{m}\vec{\sigma} \end{vmatrix}, \quad (1.3)$$

for definiteness we will use the Weyl spinor basis; four 4-dimensional vectors  $(k, m, l, n)$  are definite linear combinations of  $A, B, A_l, B_l, F_{mn}$  – see (2.4). In such parameters (2.3), the group multiplication law  $G'' = G'G$  is found in explicit form.

Then we turn to the following problem: at given  $G = G(k, m, n, l)$  one should find parameters of the inverse matrix:  $G^{-1} = G(k', m', n', l')$  – expressions for  $(k', m', n', l')$  have been found explicitly (for details of calculation see [62]). Also, several equivalent expressions for determinant  $\det G$  have been obtained, which is essential when going to special groups  $SL(4, C)$  and its subgroups.

In Section 3, with the help of the expression for the inverse matrix  $G^{-1}(k', m', l', n')$  we begin to consider the unitary group  $SU(4)$ . To this end, one should specify the requirement of unitarity  $G^+ = G^{-1}$  to the above vector parametrization – so that unitarity conditions are given as non-linear cubic algebraic equations for parameters  $(k, m, l, n)$  including complex conjugation.

In Section 4 we have constructed three 2-parametric solutions of the produced equations of unitarity<sup>4</sup>, these subgroups  $G_1, G_2, G_3$  consist of two commuting Abelian unitary subgroups.

In Section 5 we have constructed a 4-parametric solution<sup>5</sup> – it may be factorized into two commuting unitary factors:  $G = G_0 \otimes SU(2)$  – see (5.15).

The task of complete solving of the unitarity conditions seems to be rather complicated. In remaining part of the present paper we describe some relations of the above treatment to other considerations of the problem in the literature. We hope that the full general solution of the unitary equations obtained can be constructed on the way of combining different techniques used in the theory of the unitary group  $SU(4)$  and it will be considered elsewhere.

We turn again to the explicit form of the Dirac basis and note that all 15 matrices are of Gell-Mann type: they have a zero-trace, they are Hermitian, besides their squares are unite:

$$\text{Sp}\Lambda = 0, \quad (\Lambda)^2 = I, \quad (\Lambda)^+ = \Lambda, \quad \Lambda \in \{\Lambda_k, k = 1, \dots, 15\}.$$

Exponential function of any of them equals to

$$U_j = e^{ia_j\Lambda_j} = \cos a_j + i \sin a_j \Lambda_j, \quad \det e^{ia_j\Lambda_j} = +1, \quad U_j^+ = U_j^{-1}, \quad a_i \in R.$$

Evidently, multiplying such 15 elementary unitary matrices (at real parameters  $x_i$ ) gives again an unitary matrix

$$U = e^{ia_1\Lambda_1} e^{ia_2\Lambda_2} \dots e^{ia_{14}\Lambda_{14}} e^{ia_{15}\Lambda_{15}}, \quad U^+ = e^{-ia_{15}\Lambda_{15}} e^{-ia_{14}\Lambda_{14}} \dots e^{-ia_1\Lambda_1}.$$

<sup>3</sup>That Dirac matrices-based approach was widely used in physical context (see [5, 8, 9, 6, 45, 48, 49, 50, 66] and especially [40]).

<sup>4</sup>At this, the unitarity equations may be considered as special eigenvalue problems in 2-dimensional space.

<sup>5</sup>The problem again is reduced to solving of a special eigenvalue problem in 2-dimensional space.

At this there arises one special possibility to determine extended Euler angles  $a_1, \dots, a_{15}$ . For the group  $SU(4)$  the Euler parametrization of that type was found in [69]. A method to solve the problem in [69] was based on the use yet known Euler parametrization for  $SU(3)$  – the latter problem was solved in [25]. Extension to  $SU(N)$  group was done in [70, 71]. Evident advantage of the Euler angles approach is its simplicity, and evident defect consists in the following: we do not know any simple group multiplication rule for these angles – even the known solution for  $SU(2)$  is too complicated and cannot be used effectively in calculation.

In Section 6 the main question is how in Dirac parametrization one can distinguish  $SU(3)$ , the subgroup in  $SU(4)$ . In this connection, it should be noted that the basis  $\lambda_i$  used in [25] substantially differs from the above Dirac basis  $\Lambda_i$  – this peculiarity is closely connected with distinguishing the  $SU(3)$  in  $SU(4)$ . In order to have possibility to compare two approaches we need exact connection between  $\lambda_i$  and  $\Lambda_i$  – we have found required formulas<sup>6</sup>. The separation of  $SL(3, C)$  in  $SL(4, C)$  is given explicitly, at this  $3 \times 3$  matrix group is described with the help of  $4 \times 4$  matrices<sup>7</sup>. The group law for parameters of  $SL(3, C)$  is specified.

In Section 7 one different way to list 15 generators of  $GL(4, C)$  is examined<sup>8</sup>

$$\alpha_1 = \gamma^0 \gamma^2, \quad \alpha_2 = i \gamma^0 \gamma^5, \quad \alpha_3 = \gamma^5 \gamma^2, \quad \beta_1 = i \gamma^3 \gamma^1, \quad \beta_2 = i \gamma^3, \quad \beta_3 = i \gamma^1,$$

these two set commute with each others  $\alpha_j \beta_k = \beta_k \alpha_j$ , and their multiplications provides us with 9 remaining basis elements of fifteen:

$$\begin{aligned} A_1 &= \alpha_1 \beta_1, & B_1 &= \alpha_1 \beta_2, & C_1 &= \alpha_1 \beta_3, \\ A_2 &= \alpha_2 \beta_1, & B_2 &= \alpha_2 \beta_2, & C_2 &= \alpha_2 \beta_3, \\ A_3 &= \alpha_3 \beta_1, & B_3 &= \alpha_3 \beta_2, & C_3 &= \alpha_3 \beta_3. \end{aligned}$$

We turn to the rule of multiplying 15 generators  $\alpha_i, \beta_i, A_i, B_i, C_i$  and derive its explicit form (see (7.3)).

Section 8 adds some facts to a factorized structure of  $SU(4)$ . To this end, between 9 generators we distinguish three sets of commuting ones

$$\mathbf{K} = \{A_1, B_2, C_3\}, \quad \mathbf{L} = \{C_1, A_2, B_3\}, \quad \mathbf{M} = \{B_1, C_2, A_3\},$$

an arbitrary element from  $GL(4, C)$  can be factorized as follows<sup>9</sup>

$$S = e^{i\vec{a}\vec{\alpha}} e^{i\vec{b}\vec{\beta}} e^{i\mathbf{k}\mathbf{K}} e^{i\mathbf{l}\mathbf{L}} e^{i\mathbf{m}\mathbf{M}}, \quad (1.4)$$

where  $\mathbf{K}, \mathbf{L}, \mathbf{M}$  are 3-parametric groups, each of them consists of three Abelian commuting unitary subgroups<sup>10</sup>. On the basis of 15 matrices one can easily see 20 ways to separate  $SU(2)$  subgroups, which might be used as bigger elementary blocks in constructing a general transformation<sup>11</sup>.

In Sections 9 and 10 we specify our approach for pseudounitary groups  $SU(2, 2)$  and  $SU(3, 1)$  respectively. All generators  $\Lambda'_k$  of these groups can readily be constructed on the basis of the known Dirac generators of  $SU(4)$  (see (9.1)).

<sup>6</sup>This problem evidently is related to the task of distinguishing  $GL(3, C)$  in  $GL(4, C)$  as well.

<sup>7</sup>Interesting arguments related to this point but in the quaternion approach are given in [35].

<sup>8</sup>Such a possibility is well-known – see [40]; our approach looks simpler and more symmetrical because we use the Weyl basis for Dirac matrices instead of the standard one as in [40].

<sup>9</sup>These facts were described in main parts in [40].

<sup>10</sup>Note that existence of three Abelian commuting unitary subgroups was shown in [40] as well.

<sup>11</sup>This possibility was studied partly in [13, 14] on the basis of the Hamilton–Cayley approach.



## 2 On parameters of inverse transformations $G^{-1}$

Arbitrary  $4 \times 4$  matrix  $G \in GL(4, C)$  can be decomposed in terms of 16 Dirac matrices (such an approach to the group  $L(4, C)$  was discussed and partly developed in [5, 8, 9, 6, 45, 48, 49, 50, 66] and especially in [40]):

$$G = AI + iB\gamma^5 + iA_l\gamma^l + B_l\gamma^l\gamma^5 + F_{mn}\sigma_{mn}, \quad (2.1)$$

where

$$\begin{aligned} \gamma^a\gamma^b + \gamma^b\gamma^a &= 2g^{ab}, & \gamma^5 &= -i\gamma^0\gamma^1\gamma^2\gamma^3, \\ \sigma^{ab} &= \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a), & g^{ab} &= \text{diag}(+1, -1, -1, -1). \end{aligned}$$

Taking 16 coefficients  $A, B, A_l, B_l, F_{mn}$  as parameters in the group  $G = G(A, B, A_l, B_l, F_{mn})$  one can establish the corresponding multiplication law for these parameters:

$$\begin{aligned} G' &= A'I + iB'\gamma^5 + iA'_l\gamma^l + B'_l\gamma^l\gamma^5 + F'_{mn}\sigma_{mn}, \\ G &= AI + iB\gamma^5 + iA_l\gamma^l + B_l\gamma^l\gamma^5 + F_{mn}\sigma_{mn}, \\ G'' &= G'G = A''I + iB''\gamma^5 + iA''_l\gamma^l + B''_l\gamma^l\gamma^5 + F''_{mn}\sigma_{mn}, \end{aligned}$$

where

$$\begin{aligned} A'' &= A'A - B'B - A'_lA^l - B'_lB^l - \frac{1}{2}F'_{kl}F^{kl}, \\ B'' &= A'B + B'A + A'_lB^l - B'_lA^l + \frac{1}{4}F'_{mn}F_{cd}\epsilon^{mncd}, \\ A''_l &= A'A_l - B'B_l + A'_lA + B'_lB + A'^kF_{kl} \\ &\quad + F'_{lk}A^k + \frac{1}{2}B'_kF_{mn}\epsilon_l^{kmn} + \frac{1}{2}F'_{mn}B_k\epsilon_l^{mnk}, \\ B''_l &= A'B_l + B'A_l - A'_lB + B'_lA + B'^kF_{kl} \\ &\quad + F'_{lk}B^k + \frac{1}{2}A'_kF_{mn}\epsilon_l^{kmn} + \frac{1}{2}F'_{mn}A_k\epsilon_l^{mnk}, \\ F''_{mn} &= A'F_{mn} + F'_{mn}A - (A'_m A_n - A'_n A_m) - (B'_m B_n - B'_n B_m) \\ &\quad + A'_l B_k \epsilon^{lkmn} - B'_l A_k \epsilon^{lkmn} + \frac{1}{2}B'F_{kl}\epsilon^{kl}_{mn} + \frac{1}{2}F'_{kl}B\epsilon^{kl}_{mn} + (F'_{mk}F^k_n - F'_{nk}F^k_m). \end{aligned} \quad (2.2)$$

The latter formulas are correct in any basis for Dirac matrices. Below we will use mainly Weyl spinor basis:

$$\gamma^a = \begin{vmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix}, \quad \sigma^a = (I, \sigma^j), \quad \bar{\sigma}^a = (I, -\sigma^j), \quad \gamma^5 = \begin{vmatrix} -I & 0 \\ 0 & +I \end{vmatrix}.$$

With this choice, let us make 3 + 1-splitting in all the formulas:

$$G \in GL(4, C), \quad G = \begin{vmatrix} k_0 + \mathbf{k}\vec{\sigma} & n_0 - \mathbf{n}\vec{\sigma} \\ -l_0 - \mathbf{l}\vec{\sigma} & m_0 - \mathbf{m}\vec{\sigma} \end{vmatrix}, \quad (2.3)$$

where complex 4-vector parameters  $(k, l, m, n)$  are defined by [18]:

$$\begin{aligned} k_0 &= A - iB, & k_j &= a_j - ib_j, & m_0 &= A + iB, & m_j &= a_j + ib_j, \\ l_0 &= B_0 - iA_0, & l_j &= B_j - iA_j, & n_0 &= B_0 + iA_0, & n_j &= B_j + iA_j. \end{aligned} \quad (2.4)$$

For such parameters (2.3), the composition rule (2.2) will look as follows:

$$(k'', m''; n'', l'') = (k', m'; n', l')(k, m; n, l),$$

$$\begin{aligned}
k_0'' &= k_0' k_0 + \mathbf{k}' \mathbf{k} - n_0' l_0 + \mathbf{n}' l, \\
\mathbf{k}'' &= k_0' \mathbf{k} + \mathbf{k}' k_0 + i \mathbf{k}' \times \mathbf{k} - n_0' \mathbf{l} + \mathbf{n}' l_0 + i \mathbf{n}' \times \mathbf{l}, \\
m_0'' &= m_0' m_0 + \mathbf{m}' \mathbf{m} - l_0' n_0 + \mathbf{l}' \mathbf{n}, \\
\mathbf{m}'' &= m_0' \mathbf{m} + \mathbf{m}' m_0 - i \mathbf{m}' \times \mathbf{m} - l_0' \mathbf{n} + \mathbf{l}' n_0 - i \mathbf{l}' \times \mathbf{n}, \\
n_0'' &= k_0' n_0 - \mathbf{k}' \mathbf{n} + n_0' m_0 + \mathbf{n}' \mathbf{m}, \\
\mathbf{n}'' &= k_0' \mathbf{n} - \mathbf{k}' n_0 + i \mathbf{k}' \times \mathbf{n} + n_0' \mathbf{m} + \mathbf{n}' m_0 - i \mathbf{n}' \times \mathbf{m}, \\
l_0'' &= l_0' k_0 + \mathbf{l}' \mathbf{k} + m_0' l_0 - \mathbf{m}' l, \\
\mathbf{l}'' &= l_0' \mathbf{k} + \mathbf{l}' k_0 + i \mathbf{l}' \times \mathbf{k} + m_0' \mathbf{l} - \mathbf{m}' l_0 - i \mathbf{m}' \times \mathbf{l}.
\end{aligned} \tag{2.5}$$

Now let us turn to the following problem: with given  $G = G(k, m, n, l)$  one should find parameters of the inverse matrix:  $G^{-1} = G(k', m', n', l')$ . In other words, starting from

$$G(k, m, n, l) = \begin{vmatrix} +(k_0 + k_3) & +(k_1 - ik_2) & +(n_0 - n_3) & -(n_1 - in_2) \\ +(k_1 + ik_2) & +(k_0 - k_3) & -(n_1 + in_2) & +(n_0 + n_3) \\ -(l_0 + l_3) & -(l_1 - il_2) & +(m_0 - m_3) & -(m_1 - im_2) \\ -(l_1 + il_2) & -(l_0 - l_3) & -(m_1 + im_2) & +(m_0 + m_3) \end{vmatrix}, \tag{2.6}$$

one should calculate parameters of the inverse matrix  $G^{-1}$ . The problem turns to be rather complicated<sup>12</sup>, the final result is ( $D = \det G$ ,  $(mn) \equiv m_0 n_0 - \mathbf{m} \mathbf{n}$ , and so on)

$$\begin{aligned}
k_0' &= D^{-1} [k_0(mm) + m_0(ln) + l_0(nm) - n_0(lm) + i\mathbf{l}(\mathbf{m} \times \mathbf{n})], \\
\mathbf{k}' &= D^{-1} [-\mathbf{k}(mm) - \mathbf{m}(ln) - \mathbf{l}(nm) + \mathbf{n}(lm) + 2\mathbf{l} \times (\mathbf{n} \times \mathbf{m}) \\
&\quad + im_0(\mathbf{n} \times \mathbf{l}) + il_0(\mathbf{n} \times \mathbf{m}) + in_0(\mathbf{l} \times \mathbf{m})], \\
m_0' &= D^{-1} [k_0(ln) + m_0(kk) - l_0(kn) + n_0(lk) + i\mathbf{n}(\mathbf{l} \times \mathbf{k})], \\
\mathbf{m}' &= D^{-1} [-\mathbf{k}(ln) - \mathbf{m}(kk) + \mathbf{l}(kn) - \mathbf{n}(kl) + 2\mathbf{n} \times (\mathbf{l} \times \mathbf{k}) \\
&\quad + in_0(\mathbf{k} \times \mathbf{l}) + il_0(\mathbf{k} \times \mathbf{n}) + ik_0(\mathbf{n} \times \mathbf{l})], \\
l_0' &= D^{-1} [+k_0(ml) - m_0(kl) - l_0(km) - n_0(ll) + i\mathbf{m}(\mathbf{l} \times \mathbf{k})], \\
\mathbf{l}' &= D^{-1} [+ \mathbf{k}(ml) - \mathbf{m}(kl) - \mathbf{l}(km) - \mathbf{n}(ll) + 2\mathbf{m} \times (\mathbf{k} \times \mathbf{l}) \\
&\quad + im_0(\mathbf{l} \times \mathbf{k}) + ik_0(\mathbf{l} \times \mathbf{m}) + il_0(\mathbf{m} \times \mathbf{k})], \\
n_0' &= D^{-1} [-k_0(nm) + m_0(kn) - l_0(nn) - n_0(km) + i\mathbf{k}(\mathbf{m} \times \mathbf{n})], \\
\mathbf{n}' &= D^{-1} [-\mathbf{k}(nm) + \mathbf{m}(kn) - \mathbf{l}(nn) - \mathbf{n}(km) + 2\mathbf{k} \times (\mathbf{m} \times \mathbf{n}) \\
&\quad + ik_0(\mathbf{m} \times \mathbf{n}) + im_0(\mathbf{k} \times \mathbf{n}) + in_0(\mathbf{m} \times \mathbf{k})].
\end{aligned} \tag{2.7}$$

Substituting equations (2.7) into equation  $G^{-1}G = I$  one arrives at

$$\begin{aligned}
D &= k_0'' = k_0' k_0 + \mathbf{k}' \mathbf{k} - n_0' l_0 + \mathbf{n}' l, \\
0 &= \mathbf{k}'' = k_0' \mathbf{k} + \mathbf{k}' k_0 + i \mathbf{k}' \times \mathbf{k} - n_0' \mathbf{l} + \mathbf{n}' l_0 + i \mathbf{n}' \times \mathbf{l}, \\
D &= m_0'' = m_0' m_0 + \mathbf{m}' \mathbf{m} - l_0' n_0 + \mathbf{l}' \mathbf{n}, \\
0 &= \mathbf{m}'' = m_0' \mathbf{m} + \mathbf{m}' m_0 - i \mathbf{m}' \times \mathbf{m} - l_0' \mathbf{n} + \mathbf{l}' n_0 - i \mathbf{l}' \times \mathbf{n}, \\
0 &= n_0'' = k_0' n_0 - \mathbf{k}' \mathbf{n} + n_0' m_0 + \mathbf{n}' \mathbf{m}, \\
0 &= \mathbf{n}'' = k_0' \mathbf{n} - \mathbf{k}' n_0 + i \mathbf{k}' \times \mathbf{n} + n_0' \mathbf{m} + \mathbf{n}' m_0 - i \mathbf{n}' \times \mathbf{m}, \\
0 &= l_0'' = l_0' k_0 + \mathbf{l}' \mathbf{k} + m_0' l_0 - \mathbf{m}' l, \\
0 &= \mathbf{l}'' = l_0' \mathbf{k} + \mathbf{l}' k_0 + i \mathbf{l}' \times \mathbf{k} + m_0' \mathbf{l} - \mathbf{m}' l_0 - i \mathbf{m}' \times \mathbf{l}.
\end{aligned}$$

<sup>12</sup>For more details see [62]; also see a preceding paper [41].



After calculation, one can prove these identities and find the determinant:

$$\begin{aligned} D = \det G(k, m, n, l) &= (kk)(mm) + (ll)(nn) + 2(mk)(ln) + 2(lk)(nm) - 2(nk)(lm) \\ &\quad + 2i[k_0 \mathbf{l}(\mathbf{m} \times \mathbf{n}) + m_0 \mathbf{k}(\mathbf{n} \times \mathbf{l}) + l_0 \mathbf{k}(\mathbf{n} \times \mathbf{m}) + n_0 \mathbf{l}(\mathbf{m} \times \mathbf{k})] \\ &\quad + 4(\mathbf{k}\mathbf{n})(\mathbf{m}\mathbf{l}) - 4(\mathbf{k}\mathbf{m})(\mathbf{n}\mathbf{l}). \end{aligned} \quad (2.8)$$

Let us specify several more simple subgroups.

### Case A

Let 0-components  $k_0, m_0, l_0, n_0$  be real-valued, and 3-vectors  $\mathbf{k}, \mathbf{m}, \mathbf{l}, \mathbf{n}$  be imaginary. Performing in (2.5) the formal change (new vectors are real-valued)

$$\begin{aligned} \mathbf{k} &\Rightarrow i\mathbf{k}, & \mathbf{m} &\Rightarrow i\mathbf{m}, & \mathbf{l} &\Rightarrow i\mathbf{l}, & \mathbf{n} &\Rightarrow i\mathbf{n}, \\ G &= \begin{vmatrix} k_0 + i\mathbf{k}\vec{\sigma} & n_0 - i\mathbf{n}\vec{\sigma} \\ -l_0 - i\mathbf{l}\vec{\sigma} & m_0 - i\mathbf{m}\vec{\sigma} \end{vmatrix}, \end{aligned} \quad (2.9)$$

then the multiplication rules (2.5) for sixteen real variables look as follows

$$\begin{aligned} k_0'' &= k_0' k_0 - \mathbf{k}' \mathbf{k} - n_0' l_0 - \mathbf{n}' \mathbf{l}, \\ \mathbf{k}'' &= k_0' \mathbf{k} + \mathbf{k}' k_0 - \mathbf{k}' \times \mathbf{k} - n_0' \mathbf{l} + \mathbf{n}' l_0 - \mathbf{n}' \times \mathbf{l}, \\ m_0'' &= m_0' m_0 - \mathbf{m}' \mathbf{m} - l_0' n_0 - \mathbf{l}' \mathbf{n}, \\ \mathbf{m}'' &= m_0' \mathbf{m} + \mathbf{m}' m_0 + \mathbf{m}' \times \mathbf{m} - l_0' \mathbf{n} + \mathbf{l}' n_0 + \mathbf{l}' \times \mathbf{n}, \\ n_0'' &= k_0' n_0 + \mathbf{k}' \mathbf{n} + n_0' m_0 - \mathbf{n}' \mathbf{m}, \\ \mathbf{n}'' &= k_0' \mathbf{n} - \mathbf{k}' n_0 - \mathbf{k}' \times \mathbf{n} + n_0' \mathbf{m} + \mathbf{n}' m_0 + \mathbf{n}' \times \mathbf{m}, \\ l_0'' &= l_0' k_0 - \mathbf{l}' \mathbf{k} + m_0' l_0 + \mathbf{m}' \mathbf{l}, \\ \mathbf{l}'' &= l_0' \mathbf{k} + \mathbf{l}' k_0 - \mathbf{l}' \times \mathbf{k} + m_0' \mathbf{l} - \mathbf{m}' l_0 + \mathbf{m}' \times \mathbf{l}. \end{aligned}$$

Correspondingly, expression for determinant (2.8) becomes

$$\begin{aligned} D &= [kk][mm] + [ll][nn] + 2[mk][ln] + 2[lk][nm] - 2[nk][lm] \\ &\quad + 2[k_0 \mathbf{l}(\mathbf{m} \times \mathbf{n}) + m_0 \mathbf{k}(\mathbf{n} \times \mathbf{l}) + l_0 \mathbf{k}(\mathbf{n} \times \mathbf{m}) + n_0 \mathbf{l}(\mathbf{m} \times \mathbf{k})] \\ &\quad + 4(\mathbf{k}\mathbf{n})(\mathbf{m}\mathbf{l}) - 4(\mathbf{k}\mathbf{m})(\mathbf{n}\mathbf{l}), \end{aligned}$$

where the notation is used:  $[ab] = a_0 b_0 + \mathbf{a}\mathbf{b}$ .

### Case B

Equations (2.5) permit the following restrictions:

$$m_a = k_a^*, \quad l_a = n_a^*,$$

and become

$$\begin{aligned} k_0'' &= k_0' k_0 + \mathbf{k}' \mathbf{k} - n_0' n_0^* + \mathbf{n}' \mathbf{n}^*, \\ \mathbf{k}'' &= k_0' \mathbf{k} + \mathbf{k}' k_0 + i\mathbf{k}' \times \mathbf{k} - n_0' \mathbf{n}^* + \mathbf{n}' n_0^* + i\mathbf{n}' \times \mathbf{n}^*, \\ n_0'' &= k_0' n_0 - \mathbf{k}' \mathbf{n} + n_0' k_0^* + \mathbf{n}' \mathbf{k}^*, \\ \mathbf{n}'' &= k_0' \mathbf{n} - \mathbf{k}' n_0 + i\mathbf{k}' \times \mathbf{n} + n_0' \mathbf{k}^* + \mathbf{n}' k_0^* - i\mathbf{n}' \times \mathbf{k}^*. \end{aligned}$$

Determinant  $D$  is given by

$$\begin{aligned} D &= (kk)(kk)^* + (nn)^*(nn) + 2(k^*k)(n^*n) + 2(n^*k)(nk^*) - 2(nk)(nk)^* \\ &\quad + 2i[k_0 \mathbf{k}^*(\mathbf{n} \times \mathbf{n}^*) - k_0^* \mathbf{k}(\mathbf{n}^* \times \mathbf{n}) + n_0^* \mathbf{n}(\mathbf{k} \times \mathbf{k}^*) - n_0 \mathbf{n}^*(\mathbf{k}^* \times \mathbf{k})] \\ &\quad + 4(\mathbf{k}\mathbf{n})(\mathbf{k}^* \mathbf{n}^*) - 4(\mathbf{k}\mathbf{k}^*)(\mathbf{n}\mathbf{n}^*). \end{aligned}$$

### Case C

In (2.9) one can impose additional restrictions

$$m_0 = k_0, \quad l_0 = n_0, \quad \mathbf{m} = -\mathbf{k}, \quad \mathbf{l} = -\mathbf{n}; \quad (2.10)$$

at this  $G(k_0, \mathbf{k}, n_0, \mathbf{n})$  looks

$$G = \begin{vmatrix} (k_0 + i\mathbf{k}\vec{\sigma}) & (n_0 - i\mathbf{n}\vec{\sigma}) \\ -(n_0 - i\mathbf{n}\vec{\sigma}) & (k_0 + i\mathbf{k}\vec{\sigma}) \end{vmatrix};$$

and the composition rule is

$$\begin{aligned} k_0'' &= k_0'k_0 - \mathbf{k}'\mathbf{k} - n_0'n_0 + \mathbf{n}'\mathbf{n}, \\ \mathbf{k}'' &= k_0'\mathbf{k} + \mathbf{k}'k_0 - \mathbf{k}' \times \mathbf{k} + n_0'\mathbf{n} + \mathbf{n}'n_0 + \mathbf{n}' \times \mathbf{n}, \\ n_0'' &= k_0'n_0 + \mathbf{k}'\mathbf{n} + n_0'k_0 + \mathbf{n}'\mathbf{k}, \\ \mathbf{n}'' &= k_0'\mathbf{n} - \mathbf{k}'n_0 - \mathbf{k}' \times \mathbf{n} - n_0'\mathbf{k} + \mathbf{n}'k_0 - \mathbf{n}' \times \mathbf{k}. \end{aligned}$$

Determinant equals to

$$\begin{aligned} \det G &= [kk][kk] + [nn][nn] + 2(kk)(nn) + 2(nk)(nk) - 2[nk][nk] \\ &\quad + 4(\mathbf{k}\mathbf{n})(\mathbf{k}\mathbf{n}) - 4(\mathbf{k}\mathbf{k})(\mathbf{n}\mathbf{n}). \end{aligned}$$

### Case D

There exists one other subgroup defined by

$$n_a = 0, \quad l_a = 0, \quad G = \begin{vmatrix} (k_0 + \mathbf{k}\vec{\sigma}) & 0 \\ 0 & (m_0 - \mathbf{m}\vec{\sigma}) \end{vmatrix},$$

the composition law (2.5) becomes simpler

$$\begin{aligned} k_0'' &= k_0'k_0 + \mathbf{k}'\mathbf{k}, \quad \mathbf{k}'' = k_0'\mathbf{k} + \mathbf{k}'k_0 + i\mathbf{k}' \times \mathbf{k}, \\ m_0'' &= m_0'm_0 + \mathbf{m}'\mathbf{m}, \quad \mathbf{m}'' = m_0'\mathbf{m} + \mathbf{m}'m_0 - i\mathbf{m}' \times \mathbf{m}, \end{aligned}$$

as well as the determinant  $D$

$$\det G = (kk)(mm).$$

If one additionally imposes two requirements  $(kk) = +1$ ,  $(mm) = +1$ , the Case D describes spinor covering for special complex rotation group  $SO(4, C)$ ; this most simple case was considered in detail in [18].

It should be noted that the above general expression (2.8) for determinant can be transformed to a shorter form

$$\begin{aligned} \det G &= (kk)(mm) + (nn)(ll) + 2[kn][ml] \\ &\quad - 2(k_0\mathbf{n} + n_0\mathbf{k} - i\mathbf{k} \times \mathbf{n})(m_0\mathbf{l} + l_0\mathbf{m} + i\mathbf{m} \times \mathbf{l}), \end{aligned}$$

which for the three Cases A, B, C becomes yet simpler:

$$\begin{aligned} \text{(A):} \quad \det G &= [kk][mm] + [nn][ll] + 2(kn)(ml) \\ &\quad + 2(k_0\mathbf{n} + n_0\mathbf{k} + \mathbf{k} \times \mathbf{n})(m_0\mathbf{l} + l_0\mathbf{m} - \mathbf{m} \times \mathbf{l}), \\ \text{(B):} \quad \det G &= (kk)(k^*k^*) + (nn)(n^*n^*) + 2[kn][k^*n^*] \\ &\quad - 2(k_0\mathbf{n} + n_0\mathbf{k} - i\mathbf{k} \times \mathbf{n})(k_0^*\mathbf{n}^* + n_0^*\mathbf{k}^* + i\mathbf{k}^* \times \mathbf{n}^*), \\ \text{(C):} \quad \det G &= [kk]^2 + [nn]^2 + 2(kn)^2 - 2(k_0\mathbf{n} + n_0\mathbf{k} + \mathbf{k} \times \mathbf{n})^2. \end{aligned}$$

### 3 Unitarity condition

Now let us turn to consideration of the unitary group  $SU(4)$ . One should specify the requirement of unitarity  $G^+ = G^{-1}$  to the above vector parametrization. Taking into account the formulas

$$G^+ = \begin{vmatrix} k_0^* + \mathbf{k}^* \vec{\sigma} & -l_0^* - \mathbf{l}^* \vec{\sigma} \\ n_0^* - \mathbf{n}^* \vec{\sigma} & m_0^* - \mathbf{m}^* \vec{\sigma} \end{vmatrix}, \quad G^{-1} = \begin{vmatrix} k'_0 + \mathbf{k}' \vec{\sigma} & n'_0 - \mathbf{n}' \vec{\sigma} \\ -l'_0 - \mathbf{l}' \vec{\sigma} & m'_0 - \mathbf{m}' \vec{\sigma} \end{vmatrix}, \quad (3.1)$$

which can be represented differently

$$G^+ = G(k_0^*, \mathbf{k}^*; m_0^*, \mathbf{m}^*; -l_0^*, \mathbf{l}^*, -n_0^*, \mathbf{n}^*), \quad G^{-1} = G(k'_0, \mathbf{k}'; m'_0, \mathbf{m}'; n'_0, \mathbf{n}', l'_0, \mathbf{l}'),$$

we arrive at

$$\begin{aligned} k_0^* &= k'_0, & \mathbf{k}^* &= \mathbf{k}', & m_0^* &= m'_0, & \mathbf{m}^* &= \mathbf{m}', \\ -l_0^* &= n'_0, & \mathbf{l}^* &= \mathbf{n}', & -n_0^* &= l'_0, & \mathbf{n}^* &= \mathbf{l}'. \end{aligned} \quad (3.2)$$

With the use of expressions for parameters of the inverse matrix with additional restriction  $\det G = +1$  equations (3.2) can be rewritten as

$$\begin{aligned} k_0^* &= +k_0(mm) + m_0(ln) + l_0(nm) - n_0(lm) + i\mathbf{l}(\mathbf{m} \times \mathbf{n}), \\ m_0^* &= +m_0(kk) + k_0(nl) + n_0(lk) - l_0(nk) - i\mathbf{n}(\mathbf{k} \times \mathbf{l}), \\ \mathbf{k}^* &= -\mathbf{k}(mm) - \mathbf{m}(ln) - \mathbf{l}(nm) + \mathbf{n}(lm) + 2\mathbf{l} \times (\mathbf{n} \times \mathbf{m}) \\ &\quad + im_0(\mathbf{n} \times \mathbf{l}) + il_0(\mathbf{n} \times \mathbf{m}) + in_0(\mathbf{l} \times \mathbf{m}), \\ \mathbf{m}^* &= -\mathbf{m}(kk) - \mathbf{k}(nl) - \mathbf{n}(lk) + \mathbf{l}(nk) + 2\mathbf{n} \times (\mathbf{l} \times \mathbf{k}) \\ &\quad - ik_0(\mathbf{l} \times \mathbf{n}) - in_0(\mathbf{l} \times \mathbf{k}) - il_0(\mathbf{n} \times \mathbf{k}), \\ l_0^* &= +k_0(nm) - m_0(kn) + l_0(nn) + n_0(km) + i\mathbf{k}(\mathbf{n} \times \mathbf{m}), \\ n_0^* &= +m_0(lk) - k_0(ml) + n_0(ll) + l_0(mk) - i\mathbf{m}(\mathbf{l} \times \mathbf{k}), \\ \mathbf{l}^* &= -\mathbf{k}(nm) + \mathbf{m}(kn) - \mathbf{l}(nn) - \mathbf{n}(km) + 2\mathbf{k} \times (\mathbf{m} \times \mathbf{n}) \\ &\quad + ik_0(\mathbf{m} \times \mathbf{n}) + im_0(\mathbf{k} \times \mathbf{n}) + in_0(\mathbf{m} \times \mathbf{k}), \\ \mathbf{n}^* &= -\mathbf{m}(kl) + \mathbf{k}(ml) - \mathbf{n}(ll) - \mathbf{l}(mk) + 2\mathbf{m} \times (\mathbf{k} \times \mathbf{l}) - im_0(\mathbf{k} \times \mathbf{l}) \\ &\quad - ik_0(\mathbf{m} \times \mathbf{l}) - il_0(\mathbf{k} \times \mathbf{m}). \end{aligned} \quad (3.3)$$

Thus, the known form for parameters of the inverse matrix  $G^{-1}$  makes possible to write easily relations (3.3) representing the unitarity condition for group  $SU(4)$ . Here there are 16 equations for 16 variables; evidently, not all of them are independent.

Let us write down several simpler cases.

#### Case A

With formal change<sup>13</sup>

$$\mathbf{k} \Rightarrow i\mathbf{k}, \quad \mathbf{m} \Rightarrow i\mathbf{m}, \quad \mathbf{l} \Rightarrow i\mathbf{l}, \quad \mathbf{n} \Rightarrow i\mathbf{n}, \quad (3.4)$$

equations (3.3) give

$$\begin{aligned} k_0 &= +k_0[mm] + m_0[ln] + l_0[nm] - n_0[lm] + \mathbf{l}(\mathbf{m} \times \mathbf{n}), \\ m_0 &= +m_0[kk] + k_0[nl] + n_0[lk] - l_0[nk] - \mathbf{n}(\mathbf{k} \times \mathbf{l}), \end{aligned}$$

<sup>13</sup>Let 0-components  $k_0, m_0, l_0, n_0$  be real-valued, and 3-vectors  $\mathbf{k}, \mathbf{m}, \mathbf{l}, \mathbf{n}$  be imaginary.

$$\begin{aligned}
\mathbf{k} &= \mathbf{k}[mm] + \mathbf{m}[ln] + \mathbf{l}[nm] - \mathbf{n}[lm] + 2\mathbf{l} \times (\mathbf{n} \times \mathbf{m}) \\
&\quad + m_0(\mathbf{n} \times \mathbf{l}) + l_0(\mathbf{n} \times \mathbf{m}) + n_0(\mathbf{l} \times \mathbf{m}), \\
\mathbf{m} &= +\mathbf{m}[kk] + \mathbf{k}[nl] + \mathbf{n}[lk] - \mathbf{l}[nk] + 2\mathbf{n} \times (\mathbf{l} \times \mathbf{k}) \\
&\quad - k_0(\mathbf{l} \times \mathbf{n}) - n_0(\mathbf{l} \times \mathbf{k}) - l_0(\mathbf{n} \times \mathbf{k}), \\
l_0 &= +k_0[nm] - m_0[kn] + l_0[nn] + n_0[km] + \mathbf{k}(\mathbf{n} \times \mathbf{m}), \\
n_0 &= +m_0[lk] - k_0[ml] + n_0[ll] + l_0[mk] - \mathbf{m}(\mathbf{l} \times \mathbf{k}), \\
\mathbf{l} &= +\mathbf{k}[nm] - \mathbf{m}[kn] + \mathbf{l}[nn] + \mathbf{n}[km] + 2\mathbf{k} \times (\mathbf{m} \times \mathbf{n}) \\
&\quad + k_0(\mathbf{m} \times \mathbf{n}) + m_0(\mathbf{k} \times \mathbf{n}) + n_0(\mathbf{m} \times \mathbf{k}), \\
\mathbf{n} &= +\mathbf{m}[kl] - \mathbf{k}[ml] + \mathbf{n}[ll] + \mathbf{l}[mk] + 2\mathbf{m} \times (\mathbf{k} \times \mathbf{l}) \\
&\quad - m_0(\mathbf{k} \times \mathbf{l}) - k_0(\mathbf{m} \times \mathbf{l}) - l_0(\mathbf{k} \times \mathbf{m}).
\end{aligned}$$

Here there are 16 equations for 16 real-valued variables.

## Case B

Let

$$\begin{aligned}
m_0 &= k_0^*, & \mathbf{m} &= \mathbf{k}^*, & l_0 &= n_0^*, & \mathbf{l} &= \mathbf{n}^*, \\
k_0 &= m_0^*, & \mathbf{k} &= \mathbf{m}^*, & n_0 &= l_0^*, & \mathbf{n} &= \mathbf{l}^*,
\end{aligned}$$

or symbolically  $m = k^*$ ,  $l = n^*$ . The unitarity relations become

$$\begin{aligned}
k_0^* &= +k_0(k^*k^*) + k_0^*(n^*n) + n_0^*(nk^*) - n_0(n^*k^*) + i\mathbf{n}^*(\mathbf{k}^* \times \mathbf{n}), \\
\mathbf{k}^* &= -\mathbf{k}(k^*k^*) - \mathbf{k}^*(n^*n) - \mathbf{n}^*(nk^*) + \mathbf{n}(n^*k^*) \\
&\quad + 2\mathbf{n}^* \times (\mathbf{n} \times \mathbf{k}^*) + ik_0^*(\mathbf{n} \times \mathbf{n}^*) + in_0^*(\mathbf{n} \times \mathbf{k}^*) + in_0(\mathbf{l} \times \mathbf{m}), \\
n_0^* &= +k_0^*(n^*k) - k_0(k^*n^*) + n_0(n^*n^*) + n_0^*(k^*k) - i\mathbf{k}^*(\mathbf{n}^* \times \mathbf{k}), \\
\mathbf{n}^* &= -\mathbf{k}^*(kn^*) + \mathbf{k}(k^*n^*) - \mathbf{n}(n^*n^*) - \mathbf{n}^*(k^*k) \\
&\quad + 2\mathbf{k}^* \times (\mathbf{k} \times \mathbf{n}^*) - ik_0^*(\mathbf{k} \times \mathbf{n}^*) - ik_0(\mathbf{k}^* \times \mathbf{n}^*) - in_0^*(\mathbf{k} \times \mathbf{k}^*),
\end{aligned}$$

and 8 conjugated ones

$$\begin{aligned}
k_0 &= +k_0^*(kk) + k_0(nn^*) + n_0(n^*k) - n_0^*(nk) - i\mathbf{n}(\mathbf{k} \times \mathbf{n}^*), \\
\mathbf{k} &= -\mathbf{k}^*(kk) - \mathbf{k}(nn^*) - \mathbf{n}(n^*k) + \mathbf{n}^*(nk) \\
&\quad + 2\mathbf{n} \times (\mathbf{n}^* \times \mathbf{k}) - ik_0(\mathbf{n}^* \times \mathbf{n}) - in_0(\mathbf{n}^* \times \mathbf{k}) - in_0^*(\mathbf{n} \times \mathbf{k}), \\
n_0 &= +k_0(nk^*) - k_0^*(kn) + n_0^*(nn) + n_0(kk^*) + i\mathbf{k}(\mathbf{n} \times \mathbf{k}^*), \\
\mathbf{n} &= -\mathbf{k}(nk^*) + \mathbf{k}^*(kn) - \mathbf{n}^*(nn) - \mathbf{n}(kk^*) \\
&\quad + 2\mathbf{k} \times (\mathbf{k}^* \times \mathbf{n}) + ik_0(\mathbf{k}^* \times \mathbf{n}) + ik_0^*(\mathbf{k} \times \mathbf{n}) + in_0(\mathbf{k}^* \times \mathbf{k}).
\end{aligned}$$

It may be noted that latter relations are greatly simplified when  $n = 0$ , or when  $k = 0$ . Firstly, let us consider the case  $n = 0$ :

$$k_0^* = +k_0(k^*k^*), \quad \mathbf{k}^* = -\mathbf{k}(k^*k^*).$$

Taking in mind the identity

$$\det G = (kk)(kk)^* = +1 \quad \implies \quad (kk) = +1, \quad (kk)^* = +1,$$

we arrive at  $k_0^* = +k_0$ ,  $\mathbf{k}^* = -\mathbf{k}$ . It has sense to introduce the real-valued vector  $c_a$ :

$$k_0^* = +k_0 = c_0, \quad \mathbf{k}^* = -\mathbf{k} : \quad \mathbf{k} = i\mathbf{c},$$

then matrix  $G$  is

$$G(k, m = k^*, 0, 0) = \begin{vmatrix} c_0 + i\mathbf{c}\vec{\sigma} & 0 \\ 0 & c_0 - i\mathbf{c}\vec{\sigma} \end{vmatrix} \sim SU(2).$$

Another possibility is realized when  $k = 0$ :

$$n_0^* = +n_0(nn)^*, \quad \mathbf{n}^* = -\mathbf{n}(nn)^*.$$

With the use of identity

$$\det G = (nn)(nn)^* = +1 \quad \implies \quad (nn) = +1, \quad (nn)^* = +1,$$

we get

$$n_0^* = +n_0 = c_0, \quad \mathbf{n}^* = -\mathbf{n}, \quad \mathbf{n} \equiv i\mathbf{c},$$

Corresponding matrices  $G(0, 0, n, l = n^*)$  make up a special set of unitary matrices

$$G = \begin{vmatrix} 0 & c_0 - i\mathbf{c}\vec{\sigma} \\ -(c_0 + i\mathbf{c}\vec{\sigma}) & 0 \end{vmatrix}, \quad G^+ = \begin{vmatrix} 0 & -(c_0 - i\mathbf{c}\vec{\sigma}) \\ (c_0 + i\mathbf{c}\vec{\sigma}) & 0 \end{vmatrix}. \quad (3.5)$$

However, it must be noted that these matrices (3.5) do not provide us with any subgroup because  $G^2 = -I$ .

### Case C

Now in equations (3.4) one should take

$$m_0 = k_0, \quad l_0 = n_0, \quad \mathbf{m} = -\mathbf{k}, \quad \mathbf{l} = -\mathbf{n},$$

then

$$\begin{aligned} k_0 &= +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk], \\ \mathbf{k} &= \mathbf{k}[kk] - \mathbf{k}(nn) - \mathbf{n}(nk) - \mathbf{n}[nk] + 2\mathbf{n} \times (\mathbf{n} \times \mathbf{k}), \\ n_0 &= +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk), \\ \mathbf{n} &= -\mathbf{k}(kn) - \mathbf{k}[kn] + \mathbf{n}[nn] - \mathbf{n}(kk) + 2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}). \end{aligned} \quad (3.6)$$

## 4 2-parametric subgroups in $SU(4)$

To be certain in correctness of the produced equations of unitarity, one should try to solve them at least in several most simple particular cases. For instance, let us turn to the Case C and specify equations (3.6) for a subgroup arising when  $k = (k_0, k_1, 0, 0)$  and  $n = (n_0, n_1, 0, 0)$ :

$$\begin{aligned} k_0 &= +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk], \\ k_1 &= +k_1[kk] - k_1(nn) - n_1(nk) - n_1[nk], \\ n_0 &= +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk), \\ n_1 &= -k_1(kn) - k_1[kn] + n_1[nn] - n_1(kk), \end{aligned} \quad (4.1)$$

they are four non-linear equations for four real variables. It may be noted that equations (4.1) can be regarded as two eigenvalue problems in two dimensional space (with eigenvalue +1):

$$\begin{vmatrix} (k_0^2 + n_0^2) - 1 + (k_1^2 - n_1^2) & -2n_1k_1 \\ -2n_1k_1 & (k_0^2 + n_0^2) - 1 - (k_1^2 - n_1^2) \end{vmatrix} \begin{vmatrix} k_0 \\ n_0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix},$$

$$\left| \begin{array}{cc} (k_1^2 + n_1^2) - 1 + (k_0^2 - n_0^2) & -2n_0k_0 \\ -2n_0k_0 & (k_1^2 + n_1^2) - 1 - (k_0^2 - n_0^2) \end{array} \right| \left| \begin{array}{c} k_1 \\ n_1 \end{array} \right| = \left| \begin{array}{c} 0 \\ 0 \end{array} \right|.$$

The determinants in both problems must be equated to zero:

$$\begin{aligned} [(k_0^2 + n_0^2) - 1]^2 - (k_1^2 - n_1^2)^2 - 4n_1^2k_1^2 &= 0, \\ [(k_1^2 + n_1^2) - 1]^2 - (k_0^2 - n_0^2)^2 - 4n_0^2k_0^2 &= 0, \end{aligned}$$

or

$$[(k_0^2 + n_0^2) - 1]^2 - (k_1^2 + n_1^2)^2 = 0, \quad [(k_1^2 + n_1^2) - 1]^2 - (k_0^2 + n_0^2)^2 = 0.$$

The latter equations may be rewritten in factorized form:

$$\begin{aligned} [(k_0^2 + n_0^2) - 1 - (k_1^2 + n_1^2)][(k_0^2 + n_0^2) - 1 + (k_1^2 + n_1^2)] &= 0, \\ [(k_1^2 + n_1^2) - 1 - (k_0^2 + n_0^2)][(k_1^2 + n_1^2) - 1 + (k_0^2 + n_0^2)] &= 0. \end{aligned}$$

They have the structure:  $AC = 0$ ,  $BC = 0$ . Four different cases arise.

(1) Let  $C = 0$ , then

$$k_0^2 + n_0^2 + k_1^2 + n_1^2 = +1. \quad (4.2)$$

(2) Now, let  $A = 0$ ,  $B = 0$ , but a contradiction arises:  $A + B = 0$ ,  $A + B = -2$ .

(3)–(4) There are two simple cases:

$$A = 0, C = 0 \quad k_0^2 + n_0^2 = 1, \quad k_1 = 0, n_1 = 0, \quad (4.3)$$

$$B = 0, C = 0 \quad k_1^2 + n_1^2 = 1, \quad k_0 = 0, n_0 = 0. \quad (4.4)$$

Evidently, (4.3) and (4.4) can be regarded as particular cases of the above variant (4.2). Now, one should take into account additional relation

$$\begin{aligned} \det G &= [kk][kk] + [nn][nn] + 2(kk)(nn) \\ &\quad + 2(nk)(nk) - 2[nk][nk] + 4(\mathbf{k}\mathbf{n})(\mathbf{k}\mathbf{n}) - 4(\mathbf{k}\mathbf{k})(\mathbf{n}\mathbf{n}) = 1, \end{aligned}$$

which can be transformed to

$$\det G = (k_0^2 + k_1^2 + n_0^2 + n_1^2)^2 - 4(k_1n_0 + k_0n_1)^2 = +1. \quad (4.5)$$

Both equations (4.2) and (4.5) are to be satisfied

$$(k_0^2 + n_0^2 + k_1^2 + n_1^2) = 1, \quad (k_0^2 + k_1^2 + n_0^2 + n_1^2)^2 - 1 - 4(k_1n_0 + k_0n_1)^2 = 0,$$

from where it follows

$$k_1n_0 + k_0n_1 = 0, \quad k_0^2 + n_0^2 + k_1^2 + n_1^2 = +1.$$

They specify a 2-parametric unitary subgroup in  $SU(4)$

$$\begin{aligned} G_1^+ &= G_1^{-1}, \quad \det G_1 = +1, \\ k_1n_0 + k_0n_1 &= 0, \quad k_0^2 + n_0^2 + k_1^2 + n_1^2 = +1, \\ G_1 &= \begin{vmatrix} k_0 + ik_1\sigma^1 & n_0 - in_1\sigma^1 \\ -n_0 + in_1\sigma^1 & k_0 + ik_1\sigma^1 \end{vmatrix} = \begin{vmatrix} k_0 & ik_1 & n_0 & -in_1 \\ ik_1 & k_0 & -in_1 & n_0 \\ -n_0 & in_1 & k_0 & ik_1 \\ in_1 & -n_0 & ik_1 & k_0 \end{vmatrix}. \end{aligned} \quad (4.6)$$

Two analogous subgroups are possible:

$$\begin{aligned}
G_2^+ &= G_2^{-1}, \quad \det G_2 = +1, \\
k_2 n_0 + k_0 n_2 &= 0, \quad k_0^2 + n_0^2 + k_2^2 + n_2^2 = +1, \\
G_2 &= \begin{vmatrix} k_0 + ik_2\sigma^2 & n_0 - in_2\sigma^2 \\ -n_0 + in_2\sigma^2 & k_0 + ik_2\sigma^2 \end{vmatrix} = \begin{vmatrix} k_0 & k_2 & n_0 & -n_2 \\ -k_2 & k_0 & n_2 & n_0 \\ -n_0 & n_2 & k_0 & k_2 \\ -n_2 & -n_0 & -k_2 & k_0 \end{vmatrix}; \tag{4.7}
\end{aligned}$$

$$\begin{aligned}
G_3^+ &= G_3^{-1}, \quad \det G_3 = +1, \\
k_3 n_0 + k_0 n_3 &= 0, \quad k_0^2 + n_0^2 + k_3^2 + n_3^2 = +1, \\
G_3 &= \begin{vmatrix} (k_0 + ik_3) & 0 & (n_0 - in_3) & 0 \\ 0 & (k_0 - ik_3) & 0 & (n_0 + in_3) \\ -(n_0 - in_3) & 0 & (k_0 + ik_3) & 0 \\ 0 & -(n_0 + in_3) & 0 & (k_0 - ik_3) \end{vmatrix}. \tag{4.8}
\end{aligned}$$

Let us consider the latter subgroup (4.8) in some detail. The multiplication law for parameters is

$$\begin{aligned}
k_0'' &= k_0' k_0 - k_3' k_3 - n_0' n_0 + n_3' n_3, & k_3'' &= k_0' k_3 + k_3' k_0 + n_0' n_3 + n_3' n_0, \\
n_0'' &= k_0' n_0 + k_3' n_3 + n_0' k_0 + n_3' k_3, & n_3'' &= k_0' n_3 - k_3' n_0 - n_0' k_3 + n_3' k_0.
\end{aligned}$$

For two particular cases (see (4.3) and (4.4)), these formulas take the form:

$$\begin{aligned}
\{G_3^{0'}\} : & \quad k_3^2 + n_3^2 = 1, & k_0 &= 0, & n_0 &= 0, \\
& \quad k_0'' = -k_3' k_3 + n_3' n_3, & k_3'' &= 0, \\
& \quad n_0'' = +k_3' n_3 + n_3' k_3, & n_3'' &= 0, \\
\{G^0\} : & \quad k_0^2 + n_0^2 = 1, & k_3 &= 0, & n_3 &= 0, \\
& \quad k_0'' = k_0' k_0 - n_0' n_0, \\
& \quad n_0'' = k_0' n_0 + n_0' k_0. \tag{4.9}
\end{aligned}$$

Therefore, multiplying of any two elements from  $G_3^{0'}$  does not lead us to any element from  $G_3^{0'}$ , instead belonging to  $G^0$ :  $G_3^{0'} G_3^{0'} \in G^0$ . Similar result would be achieved for  $G_1$  and  $G_2$ :  $G_1^{0'} G_1^{0'} \in G^0$ ,  $G_2^{0'} G_2^{0'} \in G^0$ . In the subgroup given by (4.9) one can easily see the structure of the 1-parametric Abelian subgroup:

$$\begin{aligned}
k_0 &= \cos \alpha, & n_0 &= \sin \alpha, \\
G^0(\alpha) &= \begin{vmatrix} \cos \alpha & 0 & \sin \alpha & 0 \\ 0 & \cos \alpha & 0 & \sin \alpha \\ -\sin \alpha & 0 & \cos \alpha & 0 \\ 0 & -\sin \alpha & 0 & \cos \alpha \end{vmatrix}, & \alpha &\in [0, 2\pi]. \tag{4.10}
\end{aligned}$$

In the same manner, similar curvilinear parametrization can be readily produced for 2-parametric groups (4.6)–(4.8). For definiteness, for the subgroup  $G_3$  such coordinates are given by

$$\begin{aligned}
k_0 &= \cos \alpha \cos \rho, & k_3 &= \cos \alpha \sin \rho, \\
n_0 &= \sin \alpha \cos \rho, & -n_3 &= \sin \alpha \sin \rho, \alpha \in [0, 2\pi],
\end{aligned}$$



and matrix  $G_3$  is

$$G_3(\rho, \alpha) = \begin{vmatrix} \cos \alpha e^{i\rho} & 0 & \sin \alpha e^{i\rho} & 0 \\ 0 & \cos \alpha e^{-i\rho} & 0 & \sin \alpha e^{-i\rho} \\ -\sin \alpha e^{i\rho} & 0 & \cos \alpha e^{i\rho} & 0 \\ 0 & -\sin \alpha e^{-i\rho} & 0 & \cos \alpha e^{-i\rho} \end{vmatrix}. \quad (4.11)$$

One may note that equation (4.11) at  $\rho = 0$  will coincide with  $G^0(\alpha)$  in (4.10):  $G_3(\rho = 0, \alpha) = G^0(\alpha)$ . Similar curvilinear parametrization may be introduced for two other subgroups,  $G_1$  and  $G_2$ .

One could try to obtain more general result just changing real valued curvilinear coordinates on complex. However it is easily verified that it is not the case: through that change though there arise subgroups but they are not unitary. Indeed, let the matrix (4.10) be complex: then unitarity condition gives

$$\cos \alpha \cos \alpha^* + \sin \alpha \sin \alpha^* = 1, \quad -\cos \alpha \sin \alpha^* + \sin \alpha \cos \alpha^* = 0.$$

These two equations can be satisfied only by a real valued  $\alpha$ . In the same manner, the formal change  $\{G_1, G_2, G_3\} \Rightarrow \{G_1^C, G_2^C, G_3^C\}$  again provides us with non-unitary subgroups.

It should be noted that each of three 2-parametric subgroup  $G_1, G_2, G_3$ , in addition to  $G_0(\alpha)$ , contains one additional Abelian unitary subgroup:

$$\begin{aligned} K_1 &= \begin{vmatrix} k_0 + ik_1\sigma^1 & 0 \\ 0 & k_0 + ik_1\sigma^1 \end{vmatrix}, & K_1 \subset G_1, & k_0^2 + k_1^2 = 1, \\ K_2 &= \begin{vmatrix} k_0 + ik_2\sigma^2 & 0 \\ 0 & k_0 + ik_2\sigma^2 \end{vmatrix}, & K_2 \subset G_2, & k_0^2 + k_2^2 = 1, \\ K_3 &= \begin{vmatrix} k_0 + ik_3\sigma^3 & 0 \\ 0 & k_0 + ik_3\sigma^3 \end{vmatrix}, & K_3 \subset G_3, & k_0^2 + k_3^2 = 1. \end{aligned}$$

It may be easily verified that

$$G_1 = G_0 K_1 = K_1 G_0, \quad G_2 = G_0 K_2 = K_2 G_0, \quad G_3 = G_0 K_3 = K_3 G_0.$$

Indeed

$$G_0(\alpha) K_1 = K_1 G_0(\alpha) = \begin{vmatrix} \cos \alpha k_0 + i \cos \alpha k_1 \sigma^1 & \sin \alpha k_0 + i \sin \alpha k_1 \\ -\sin \alpha k_0 - i \sin \alpha k_1 & \cos \alpha k_0 + i \cos \alpha k_1 \sigma^1 \end{vmatrix},$$

and with notation

$$\begin{aligned} \cos \alpha k_0 &= k'_0, & \cos \alpha k_1 &= k'_1, & \sin \alpha k_0 &= n'_0, & \sin \alpha k_1 &= -n'_1, \\ k'_0 n'_1 + k'_1 n'_0 &= 0, & k_0'^2 + k_3'^2 + n_0'^2 + n_3'^2 &= 1 \end{aligned}$$

we arrive at

$$G_0 K_1 = K_1 G_0 = \begin{vmatrix} k'_0 + ik'_1 \sigma^1 & n'_0 - n'_1 \\ -n'_0 + in'_1 & k'_0 + ik'_1 \sigma^1 \end{vmatrix} \subset G_1.$$

## 5 4-parametric unitary subgroup

Let us turn again to the subgroup in  $GL(4, C)$  given by Case C (see (2.10)):

$$G = \begin{vmatrix} (k_0 + \mathbf{k}\vec{\sigma}) & (n_0 - \mathbf{n}\vec{\sigma}) \\ -(n_0 - \mathbf{n}\vec{\sigma}) & (k_0 + \mathbf{k}\vec{\sigma}) \end{vmatrix},$$

when the unitarity equations look as follows:

$$\begin{aligned} k_0 &= +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk], \\ n_0 &= +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk), \\ \mathbf{k} &= \mathbf{k}[kk] - \mathbf{k}(nn) - \mathbf{n}(nk) - \mathbf{n}[nk] + 2\mathbf{n} \times (\mathbf{n} \times \mathbf{k}), \\ \mathbf{n} &= -\mathbf{k}(kn) - \mathbf{k}[kn] + \mathbf{n}[nn] - \mathbf{n}(kk) + 2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}). \end{aligned}$$

They can be rewritten as four eigenvalue problems:

$$\begin{vmatrix} [kk] + (nn) & (nk) - [nk] \\ (nk) - [nk] & (kk) + [nn] \end{vmatrix} \begin{vmatrix} k_0 \\ n_0 \end{vmatrix} = (+1) \begin{vmatrix} k_0 \\ n_0 \end{vmatrix}, \quad (5.1)$$

$$\begin{aligned} &\begin{vmatrix} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{vmatrix} \begin{vmatrix} k_1 \\ n_1 \end{vmatrix} = (+1) \begin{vmatrix} k_1 \\ n_1 \end{vmatrix}, \\ &\begin{vmatrix} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{vmatrix} \begin{vmatrix} k_2 \\ n_2 \end{vmatrix} = (+1) \begin{vmatrix} k_2 \\ n_2 \end{vmatrix}, \\ &\begin{vmatrix} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{vmatrix} \begin{vmatrix} k_3 \\ n_3 \end{vmatrix} = (+1) \begin{vmatrix} k_3 \\ n_3 \end{vmatrix}. \end{aligned} \quad (5.2)$$

These equations have the same structure

$$\begin{vmatrix} A & C \\ C & B \end{vmatrix} \begin{vmatrix} Z_1 \\ Z_2 \end{vmatrix} = \lambda \begin{vmatrix} Z_1 \\ Z_2 \end{vmatrix},$$

where  $\lambda = +1$ . Non-trivial solutions may exist only if

$$\det \begin{vmatrix} A - \lambda & C \\ C & B - \lambda \end{vmatrix} = 0,$$

which gives two different eigenvalues

$$\lambda_1 = \frac{A + B + \sqrt{(A - B)^2 + 4C^2}}{2}, \quad \lambda_2 = \frac{A + B - \sqrt{(A - B)^2 + 4C^2}}{2}.$$

In explicit form, equations (5.1) looks as follows:

$$\begin{aligned} &\begin{vmatrix} A & C \\ C & B \end{vmatrix} \begin{vmatrix} k_0 \\ n_0 \end{vmatrix} = \lambda \begin{vmatrix} k_0 \\ n_0 \end{vmatrix}, \\ &A = (k_0^2 + n_0^2) + (\mathbf{k}^2 - \mathbf{n}^2), \quad B = (k_0^2 + n_0^2) - (\mathbf{k}^2 - \mathbf{n}^2), \quad C = -2\mathbf{k}\mathbf{n}, \\ &\lambda_1 = (k_0^2 + n_0^2) + \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{k}\mathbf{n})^2}, \\ &\lambda_2 = (k_0^2 + n_0^2) - \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{k}\mathbf{n})^2}. \end{aligned} \quad (5.3)$$

The eigenvalue  $\lambda = +1$  might be constructed by two ways:

$$\begin{aligned} \lambda_1 &= +1, \quad k_0^2 + n_0^2 = 1 - \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{k}\mathbf{n})^2}, \\ \lambda_2 &= +1, \quad k_0^2 + n_0^2 = 1 + \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{k}\mathbf{n})^2}. \end{aligned} \quad (5.4)$$

These two relations (5.4) are equivalent to the following one:

$$(1 - k_0^2 - n_0^2)^2 = (\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{k}\mathbf{n})^2.$$

Thus, equations (5.3) have two different types:

Type I

$$\begin{aligned} (A-1)k_0 + Cn_0 &= 0, & Ck_0 + (B-1)n_0 &= 0, \\ k_0^2 + n_0^2 &= 1 - \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{k}\mathbf{n})^2}, \\ k_0^2 + n_0^2 &< +1, & (\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{k}\mathbf{n})^2 &< +1; \end{aligned} \quad (5.5)$$

Type II

$$\begin{aligned} (A-1)k_0 + Cn_0 &= 0, & Ck_0 + (B-1)n_0 &= 0, \\ k_0^2 + n_0^2 &= 1 + \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{k}\mathbf{n})^2}, & k_0^2 + n_0^2 &> +1. \end{aligned} \quad (5.6)$$

Now let us turn to equations (5.2). They have the form

$$\begin{aligned} \begin{vmatrix} A & C \\ C & -A \end{vmatrix} \begin{vmatrix} k_i \\ n_i \end{vmatrix} &= \lambda \begin{vmatrix} k_i \\ n_i \end{vmatrix}, & i &= 1, 2, 3, \\ A &= k_0^2 + \mathbf{k}^2 - n_0^2 - \mathbf{n}^2, & B &= -A, & C &= -2(k_0n_0 - \mathbf{k}\mathbf{n}), \\ \lambda_1 &= +\sqrt{(k_0^2 + \mathbf{k}^2 - n_0^2 - \mathbf{n}^2)^2 + 4(k_0n_0 - \mathbf{k}\mathbf{n})^2}, \\ \lambda_2 &= -\sqrt{(k_0^2 + \mathbf{k}^2 - n_0^2 - \mathbf{n}^2)^2 + 4(k_0n_0 - \mathbf{k}\mathbf{n})^2}. \end{aligned}$$

As we are interested only in positive eigenvalue  $\lambda = +1$ , we must use only one possibility  $\lambda = +1 = \lambda_1$ , so that

$$\begin{aligned} (A-1)\mathbf{k} + C\mathbf{n} &= 0, & C\mathbf{k} - (A+1)\mathbf{n} &= 0, \\ 1 &= (k_0^2 + \mathbf{k}^2 - n_0^2 - \mathbf{n}^2)^2 + 4(k_0n_0 - \mathbf{k}\mathbf{n})^2. \end{aligned} \quad (5.7)$$

Vector condition in (5.7) says that  $\mathbf{k}$  and  $\mathbf{n}$  are (anti)collinear:

$$\mathbf{k} = K\mathbf{e}, \quad \mathbf{n} = N\mathbf{e}, \quad \mathbf{e}^2 = 1, \quad \mathbf{e} \in S_2, \quad (5.8)$$

so that (5.7) give

$$\begin{aligned} (A-1)K + CN &= 0, & CK - (A+1)N &= 0, \\ 1 &= (k_0^2 + K^2 - n_0^2 - N^2)^2 + 4(k_0n_0 - KN)^2, \\ A &= k_0^2 + K^2 - n_0^2 - N^2, & C &= -2(k_0n_0 - KN). \end{aligned} \quad (5.9)$$

With notation (5.8), equations (5.5)–(5.6) take the form:

Type I

$$(A-1)k_0 + Cn_0 = 0, \quad Ck_0 + (B-1)n_0 = 0, \quad k_0^2 + n_0^2 = 1 - (K^2 + N^2),$$

Type II

$$(A-1)k_0 + Cn_0 = 0, \quad Ck_0 + (B-1)n_0 = 0, \quad k_0^2 + n_0^2 = 1 + (K^2 + N^2), \quad (5.10)$$

where

$$A = (k_0^2 + n_0^2) + (K^2 - N^2), \quad B = (k_0^2 + n_0^2) - (K^2 - N^2), \quad C = -2KN. \quad (5.11)$$

Therefore, we have 8 variables  $e, k_0, n_0, K, N$  and the set of equations, (5.9)–(5.11) for them. Its solving turns to be rather involving, so let us formulate only the final result:

$$\begin{aligned} k_0, \quad \mathbf{k} = K\mathbf{e}, \quad n_0, \quad \mathbf{n} = N\mathbf{e}, \\ k_0^2 + K^2 + n_0^2 + N^2 = +1, \quad k_0N + n_0K = 0, \\ G = \begin{vmatrix} (k_0 + iK\mathbf{e}\vec{\sigma}) & (n_0 - iN\mathbf{e}\vec{\sigma}) \\ -(n_0 - iN\mathbf{e}\vec{\sigma}) & (k_0 + iK\mathbf{e}\vec{\sigma}) \end{vmatrix}. \end{aligned} \quad (5.12)$$

It should be noted that

$$\det G = (k_0^2 + K^2 + n_0^2 + N^2)^2 = +1.$$

The unitarity of the matrices (5.12) may be verified by direct calculation. Indeed,

$$G^+ = \begin{vmatrix} (k_0 - iK\mathbf{e}\vec{\sigma}) & -(n_0 + iN\mathbf{e}\vec{\sigma}) \\ (n_0 + iN\mathbf{e}\vec{\sigma}) & (k_0 - iK\mathbf{e}\vec{\sigma}) \end{vmatrix},$$

and further for  $GG^+ = I$  we get (by  $2 \times 2$  blocks)

$$\begin{aligned} (GG^+)_{11} &= k_0^2 + K^2 + n_0^2 + N^2 = +1, & (GG^+)_{12} &= -2i(n_0K + k_0N)(\mathbf{e}\vec{\sigma}) = 0, \\ (GG^+)_{22} &= k_0^2 + K^2 + n_0^2 + N^2 = +1, & (GG^+)_{21} &= +2i(n_0K + k_0N)(\mathbf{e}\vec{\sigma}) = 0. \end{aligned}$$

One different way to parameterize (5.12) can be proposed. Indeed, relations (5.12) are

$$\begin{aligned} k_0, \quad \mathbf{k} = K\mathbf{e}, \quad n_0, \quad \mathbf{n} = N\mathbf{e}, \\ k_0^2(1 + \frac{K^2}{k_0^2}) + n_0^2(1 + \frac{N^2}{n_0^2}) = +1, \quad \frac{K}{k_0} = -\frac{N}{n_0} \equiv W, \end{aligned}$$

or

$$\begin{aligned} k_0, \quad \mathbf{k} = k_0W\mathbf{e}, \quad n_0, \quad \mathbf{n} = -n_0W\mathbf{e}, \\ (k_0^2 + n_0^2)(1 + W^2) = +1, \quad K = k_0W, \quad N = -n_0W, \quad 0 \leq k_0^2 + n_0^2 \leq 1. \end{aligned}$$

Therefore, matrix  $G$  can be presented as follows:

$$G = \begin{vmatrix} k_0(1 + iW\mathbf{e}\vec{\sigma}) & n_0(1 + iW\mathbf{e}\vec{\sigma}) \\ -n_0(1 + iW\mathbf{e}\vec{\sigma}) & k_0(1 + iW\mathbf{e}\vec{\sigma}) \end{vmatrix}, \quad (5.13)$$

$$(k_0^2 + n_0^2)(1 + W^2) = +1 \quad \implies \quad W = \pm \sqrt{\frac{1}{k_0^2 + n_0^2} - 1}.$$

Evidently, it suffices to take positive values for  $W$ . The constructed subgroup (5.13) depends upon four parameters  $k_0, n_0, \mathbf{e}$ :

$$0 \leq k_0^2 + n_0^2 \leq 1, \quad \mathbf{e}^2 = 1, \quad (k_0^2 + n_0^2)(1 + W^2) = +1.$$

Let us establish the law of multiplication for four parameters  $k_0, n_0, \mathbf{W} = W\mathbf{e}$ :

$$G'' = \begin{vmatrix} k'_0(1 + i\mathbf{W}'\vec{\sigma}) & n'_0(1 + i\mathbf{W}'\vec{\sigma}) \\ -n'_0(1 + i\mathbf{W}'\vec{\sigma}) & k'_0(1 + i\mathbf{W}'\vec{\sigma}) \end{vmatrix} \begin{vmatrix} k_0(1 + i\mathbf{W}\vec{\sigma}) & n_0(1 + i\mathbf{W}\vec{\sigma}) \\ -n_0(1 + i\mathbf{W}\vec{\sigma}) & k_0(1 + i\mathbf{W}\vec{\sigma}) \end{vmatrix}$$

or by  $2 \times 2$  blocks

$$(11) = (k'_0k_0 - n'_0n_0)(1 + i\mathbf{W}'\vec{\sigma})(1 + i\mathbf{W}\vec{\sigma}),$$

$$\begin{aligned}
(12) &= (k'_0 n_0 + n'_0 k_0)(1 + i\mathbf{W}'\vec{\sigma})(1 + i\mathbf{W}\vec{\sigma}), \\
(21) &= -(k'_0 n_0 + n'_0 k_0)(1 + i\mathbf{W}'\vec{\sigma})(1 + i\mathbf{W}\vec{\sigma}), \\
(22) &= (k'_0 k_0 - n'_0 n_0)(1 + i\mathbf{W}'\vec{\sigma})(1 + i\mathbf{W}\vec{\sigma}).
\end{aligned}$$

As (11) = (22), (12) = -(21); further one can consider only two blocks:

$$\begin{aligned}
(11) &= (k'_0 k_0 - n'_0 n_0)(1 - \mathbf{W}'\mathbf{W}) \left( 1 + i \frac{\mathbf{W}' + \mathbf{W} - \mathbf{W}' \times \mathbf{W}}{1 - \mathbf{W}'\mathbf{W}} \vec{\sigma} \right), \\
(12) &= (k'_0 n_0 + n'_0 k_0)(1 - \mathbf{W}'\mathbf{W}) \left( 1 + i \frac{\mathbf{W}' + \mathbf{W} - \mathbf{W}' \times \mathbf{W}}{1 - \mathbf{W}'\mathbf{W}} \vec{\sigma} \right).
\end{aligned}$$

So the composition rules should be

$$\begin{aligned}
k''_0 &= (k'_0 k_0 - n'_0 n_0)(1 - \mathbf{W}'\mathbf{W}), & n''_0 &= (k'_0 n_0 + n'_0 k_0)(1 - \mathbf{W}'\mathbf{W}), \\
\mathbf{W}'' &= \frac{\mathbf{W}' + \mathbf{W} - \mathbf{W}' \times \mathbf{W}}{1 - \mathbf{W}'\mathbf{W}}.
\end{aligned}$$

The later formula coincides with the Gibbs multiplication rule (see in [32]) for 3-dimensional rotation group  $SO(3, R)$ . It remains to prove the identity:

$$(k''_0{}^2 + n''_0{}^2)(1 + \mathbf{W}''^2) = +1$$

which reduces to

$$(k'_0 k_0 - n'_0 n_0)^2 + (k'_0 n_0 + n'_0 k_0)^2 [(1 - \mathbf{W}'\mathbf{W})^2 + (\mathbf{W}' + \mathbf{W} - \mathbf{W}' \times \mathbf{W})^2] = 1. \quad (5.14)$$

First terms are

$$(k'_0 k_0 - n'_0 n_0)^2 + (k'_0 n_0 + n'_0 k_0)^2 = (k_0'^2 + n_0'^2)(k_0^2 + n_0^2).$$

Second term is

$$(1 - \mathbf{W}'\mathbf{W})^2 + (\mathbf{W}' + \mathbf{W} - \mathbf{W}' \times \mathbf{W})^2 = (1 + \mathbf{W}'^2)(1 + \mathbf{W}^2).$$

Therefore, (5.14) takes the form

$$(k_0'^2 + n_0'^2)(k_0^2 + n_0^2)(1 + \mathbf{W}'^2)(1 + \mathbf{W}^2) = 1$$

which is identity due to equalities

$$(k_0'^2 + n_0'^2)(1 + \mathbf{W}'^2) = 1, \quad (k_0^2 + n_0^2)(1 + \mathbf{W}^2) = 1.$$

It is matter of simple calculation to introduce curvilinear parameters for such an unitary subgroup:

$$\begin{aligned}
\mathbf{e} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \\
k_0 &= \cos \alpha \cos \rho, & K &= \cos \alpha \sin \rho, & n_0 &= \sin \alpha \cos \rho, & N &= -\sin \alpha \sin \rho,
\end{aligned}$$

and  $G$  looks as follows

$$\begin{aligned}
G &= \begin{vmatrix} \Delta & \Sigma \\ -\Sigma & \Delta \end{vmatrix}, & \Delta &= \begin{vmatrix} \cos \alpha (\cos \rho + i \sin \rho \cos \theta) & i \cos \alpha \sin \rho \sin \theta e^{-i\phi} \\ i \cos \alpha \sin \rho \sin \theta e^{i\phi} & \cos \alpha (\cos \rho - i \sin \rho \cos \theta) \end{vmatrix}, \\
\Sigma &= \begin{vmatrix} \sin \alpha (\cos \rho + i \sin \rho \cos \theta) & +i \sin \alpha \sin \rho \sin \theta e^{-i\phi} \\ i \sin \alpha \sin \rho \sin \theta e^{i\phi} & \sin \alpha (\cos \rho - i \sin \rho \cos \theta) \end{vmatrix}.
\end{aligned}$$

It should be noted that one may factorize 4-parametric element into two unitary factors, 1-parametric and 3-parametric. Indeed, let us consider the product of commuting unitary groups, isomorphic to Abelian group  $G_0$  and  $SU(2)$ :

$$\begin{aligned} G = G_0 \otimes SU(2) &= SU(2) \otimes G_0 = \begin{vmatrix} k'_0 & n'_0 \\ -n'_0 & k'_0 \end{vmatrix} \begin{vmatrix} a_0 + i\mathbf{a}\vec{\sigma} & 0 \\ 0 & a_0 + i\mathbf{a}\vec{\sigma} \end{vmatrix} \\ &= \begin{vmatrix} k'_0 a_0 + i k'_0 \mathbf{a}\vec{\sigma} & n'_0 a_0 + i n'_0 \mathbf{a}\vec{\sigma} \\ -n'_0 a_0 - i n'_0 \mathbf{a}\vec{\sigma} & k'_0 a_0 + i k'_0 \mathbf{a}\vec{\sigma} \end{vmatrix}, \quad k_0'^2 + n_0'^2 = 1, \quad a_0^2 + \mathbf{a}^2 = 1, \end{aligned}$$

with the notation

$$\begin{aligned} k'_0 a_0 &= k_0, \quad k'_0 \mathbf{a} = k_0 \mathbf{W}, \quad n'_0 a_0 = n_0, \quad n'_0 \mathbf{a} = n_0 \mathbf{W}, \\ (k_0'^2 + n_0'^2)(1 + W^2) &= (k_0'^2 a_0^2 + n_0'^2 a_0^2)(1 + W^2) = (k_0'^2 + n_0'^2)(a_0^2 + \mathbf{a}^2) = 1, \end{aligned}$$

takes the form

$$G_0 \otimes SU(2) = SU(2) \otimes G_0 = \begin{vmatrix} k_0(1 + i\mathbf{W}\vec{\sigma}) & n_0(1 + i\mathbf{W}\vec{\sigma}) \\ -n_0(1 + i\mathbf{W}\vec{\sigma}) & k_0(1 + i\mathbf{W}\vec{\sigma}) \end{vmatrix} = G. \quad (5.15)$$

Let us summarize the main results of the previous sections:

Parametrization of  $4 \times 4$  matrices  $G$  of the complex linear group  $GL(4, C)$  in terms of four complex vector-parameters  $G = G(k, m, n, l)$  is developed and the problem of inverting any  $4 \times 4$  matrix  $G$  is solved. Expression for determinant of any matrix  $G$  is found:  $\det G = F(k, m, n, l)$ . Unitarity conditions have been formulated in the form of non-linear cubic algebraic equations including complex conjugation. Several simplest solutions of these unitarity equations have been found: three 2-parametric subgroups  $G_1, G_2, G_3$  – each of subgroups consists of two commuting Abelian unitary groups; 4-parametric unitary subgroup consisting of a product of a 3-parametric group isomorphic  $SU(2)$  and 1-parametric Abelian group.

The task of full solving of the unitarity conditions seems to be rather complicated and it will be considered elsewhere. In the remaining part of the present paper we describe some relations of the above treatment to other considerations of the problem in the literature. The relations described give grounds to hope that the full general solution of the unitary equations obtained can be constructed on the way of combining different techniques used in the theory of the unitary group  $SU(4)$ .

## 6 On subgroups $GL(3, C)$ and $SU(3)$ , expressions for Gell-Mann matrices through the Dirac basis

In this section the main question is how in the Dirac parametrization one can distinguish  $GL(3, C)$ , subgroup in  $GL(4, C)$ . To this end, let us turn to the explicit form of the Dirac basis (the Weyl representation is used; at some elements the imaginary unit  $i$  is added)

$$\begin{aligned} \gamma^5 &= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad \gamma^0 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad i\gamma^5\gamma^0 = \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix}, \\ i\gamma^1 &= \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \quad \gamma^5\gamma^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad i\gamma^2 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad (6.1) \end{aligned}$$

$$\begin{aligned}
\gamma^5 \gamma^2 &= \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, & i\gamma^3 &= \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix}, & \gamma^5 \gamma^3 &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \\
2\sigma^{01} &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, & 2\sigma^{02} &= \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}, & 2\sigma^{03} &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \\
2i\sigma^{12} &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, & 2i\sigma^{23} &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, & 2i\sigma^{31} &= \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}.
\end{aligned}$$

All these 15 matrices  $\Lambda_i$  are of Gell-Mann type: they have a zero-trace, they are Hermitian, besides, their squares are unite:

$$\text{Sp } \Lambda = 0, \quad (\Lambda)^2 = I, \quad (\Lambda)^+ = \Lambda, \quad \Lambda \in \{\Lambda_k : k = 1, \dots, 15\}.$$

Exponential function of any of them equals to

$$U = e^{ia\Lambda} = \cos a + i \sin a \Lambda, \quad \det e^{ia\Lambda} = +1, \quad U^+ = U^{-1}, \quad a \in \mathbb{R}.$$

Evidently, multiplying of such 15 elementary unitary matrices (at real parameters  $x_i$ ) results in an unitary matrix

$$U = e^{ia_1\Lambda_1} e^{ia_2\Lambda_2} \dots e^{ia_{14}\Lambda_{14}} e^{ia_{15}\Lambda_{15}}.$$

At this there arise 15 generalized angle-variables  $a_1, \dots, a_{15}$ . Evident advantage of this approach is its simplicity, and evident defect consists in the following: we do not know any simple group multiplication rule for these angles.

It should be noted that the basis  $\lambda_i$  used in [69] substantially differs from the above Dirac basis  $\Lambda_i$  – this peculiarity is closely connected with distinguishing  $SU(3)$  in  $SU(4)$ . This problem is evidently related to the task of distinguishing  $GL(3, C)$  in  $GL(4, C)$  as well.

In order to have possibility to compare two approaches we need exact connection between  $\lambda_i$  and  $\Lambda_i$ . In [69] the following Gell-Mann basis for  $SU(4)$  were used:

$$\begin{aligned}
\lambda_1 &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, & \lambda_2 &= \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, & \lambda_3 &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \\
\lambda_4 &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, & \lambda_5 &= \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, & \lambda_6 &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \\
\lambda_7 &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, & \lambda_9 &= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \\
\lambda_{10} &= \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, & \lambda_{11} &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, & \lambda_{12} &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix},
\end{aligned}$$



$$\lambda_{13} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \lambda_{14} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}, \quad \lambda_{15} = \frac{1}{\sqrt{6}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix}. \quad (6.2)$$

All the  $\lambda$  excluding  $\lambda_8, \lambda_{15}$  possess the same property:

$$\lambda_i^3 = +\lambda_i, \quad i \neq 8, 15.$$

The minimal polynomials for  $\lambda_8, \lambda_{15}$  can be easily found. Indeed,

$$(\lambda_8)^2 = \frac{1}{3} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad (\lambda_8)^3 = \frac{1}{3\sqrt{3}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

therefore

$$(\lambda_8)^3 = \frac{2}{3}\lambda_8 - \frac{1}{\sqrt{3}}(\lambda_8)^2.$$

Analogously, for  $\lambda_{15}$  we have

$$(\lambda_{15})^2 = \frac{1}{6} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{vmatrix}, \quad (\lambda_{15})^3 = \frac{1}{6\sqrt{6}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -27 \end{vmatrix},$$

and

$$(\lambda_{15})^3 = \frac{1}{2}\lambda_{15} - \frac{2}{\sqrt{6}}(\lambda_{15})^2.$$

Comparing  $\Lambda_i$  and  $\lambda_i$  one can readily derive the linear combinations:

$$\begin{aligned} \gamma^0 + \gamma^5 \gamma^3 &= 2\lambda_4, & \gamma^0 - \gamma^5 \gamma^3 &= 2\lambda_{11}, & i\gamma^5 \gamma^0 + i\gamma^3 &= 2\lambda_5, \\ i\gamma^5 \gamma^0 - i\gamma^3 &= 2\lambda_{12}, & \gamma^5 \gamma^1 + i\gamma^2 &= 2\lambda_6, & \gamma^5 \gamma^1 - i\gamma^2 &= 2\lambda_9, \\ i\gamma^1 + \gamma^5 \gamma^2 &= 2\lambda_{10}, & i\gamma^1 - \gamma^5 \gamma^2 &= 2\lambda_7, & 2\sigma^{01} + 2i\sigma^{23} &= 2\lambda_1, \\ 2\sigma^{01} - 2i\sigma^{23} &= -2\lambda_{13}, & 2\sigma^{02} + 2i\sigma^{31} &= 2\lambda_2, & 2\sigma^{02} - 2i\sigma^{31} &= -2\lambda_{14}, \end{aligned}$$

and additional six combinations

$$\begin{aligned} 2\sigma^{03} + 2i\sigma^{12} &= \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, & 2\sigma^{03} - 2i\sigma^{12} &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & +2 \end{vmatrix}, \\ \gamma^5 + 2\sigma^{03} &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}, & \gamma^5 - 2\sigma^{03} &= \begin{vmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \\ \gamma^5 + 2i\sigma^{12} &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, & \gamma^5 - 2i\sigma^{12} &= \begin{vmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}, \end{aligned}$$

they should contain three linearly independent matrices. Those three linearly independent matrices might be chosen in different ways. Let us introduce the notation:

$$a = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad b = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad c = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$A = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{vmatrix}, \quad B = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad C = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

The matrices  $a, b, c$  have the  $3 \times 3$  blocks different from zero, so they could be generators for  $SU(3)$  transformations; whereas  $A, B, C$  may be generators only of the group  $SU(4)$ . All six matrices  $a, b, c, A, B, C$  have the same minimal polynomial:

$$\lambda^3 = \lambda.$$

Linear space to which these six matrices  $a, b, c, A, B, C$  belong is 3-dimensional. Indeed, one easily obtains

$$\begin{aligned} c &= b - a, & C - A &= b, & C - B &= a, & B - A &= c, \\ \text{basis } \{a, b, C\} &\implies & c &= b - a, & A &= C - b, & B &= C - a. \end{aligned}$$

These relations can be rewritten differently

$$\begin{aligned} a &= b - c, & C - A &= b, & C - B &= b - c, & B - A &= c, \\ \text{basis } \{b, c, A\} &\implies & a &= b - c, & C &= A + b, & B &= A + c \end{aligned}$$

or

$$\begin{aligned} b &= a + c, & C - A &= b, & C - B &= b - c, & B - A &= c, \\ \text{basis } \{a, c, B\} &\implies & b &= a + c, & C &= B + a, & A &= B - c. \end{aligned}$$

One should note that in the basis  $\lambda_i$  (6.2) the corresponding three linearly independent elements  $\lambda_3, \lambda_8, \lambda_{15}$  are taken as follows:

$$\begin{aligned} \lambda_3 &= \frac{1}{2}2\sigma^{03} + \frac{1}{2}2i\sigma^{12}, & \lambda_8 &= -\frac{1}{\sqrt{3}}\gamma^5 + \frac{1}{2\sqrt{3}}2\sigma^{03} - \frac{1}{2\sqrt{3}}2i\sigma^{12}, \\ \lambda_{15} &= -\frac{1}{\sqrt{6}}\gamma^5 - \frac{1}{\sqrt{6}}2\sigma^{03} + \frac{1}{\sqrt{6}}2i\sigma^{12}, \end{aligned}$$

their minimal polynomials look

$$(\lambda_3)^3 = \lambda_3, \quad (\lambda_8)^3 = \frac{2}{3}\lambda_8 - \frac{1}{\sqrt{3}}(\lambda_8)^2, \quad (\lambda_{15})^3 = \frac{1}{2}\lambda_{15} - \frac{2}{\sqrt{6}}(\lambda_{15})^2.$$

It is the matter of simple calculation to find relationships between  $\lambda_3, \lambda_8, \lambda_{15}$  and the basis  $\{a, b, C\}$ :

$$\lambda_3 = a, \quad \lambda_8 = \frac{1}{\sqrt{3}}a - \frac{2}{\sqrt{3}}b, \quad \lambda_{15} = \frac{1}{\sqrt{6}}a + \frac{1}{\sqrt{6}}b - \frac{3}{\sqrt{6}}C. \quad (6.3)$$

In the following we will use the notation

$$a = \lambda_3, \quad b = \lambda'_8, \quad C = \lambda'_{15},$$

so the previous formulas (6.3) will read  $(\{\lambda_3, \lambda_8, \lambda_{15}\} \iff \{\lambda_3, \lambda'_8, \lambda'_{15}\})$

$$\lambda_3 = \lambda_3, \quad \lambda_8 = \frac{1}{\sqrt{3}}\lambda_3 - \frac{2}{\sqrt{3}}\lambda'_8, \quad \lambda_{15} = \frac{1}{\sqrt{6}}\lambda_3 + \frac{1}{\sqrt{6}}\lambda'_8 - \frac{3}{\sqrt{6}}\lambda'_{15}.$$

The inverse relations are

$$\lambda_3 = \lambda_3, \quad \lambda'_8 = \frac{1}{2}\lambda_3 - \frac{\sqrt{3}}{2}\lambda_8, \quad \lambda'_{15} = -\frac{\sqrt{6}}{3}\lambda_{15} + \frac{1}{2}\lambda_3 - \frac{1}{2\sqrt{3}}\lambda_8. \quad (6.4)$$

Now, starting with the linear decomposition of  $G \in GL(4, C)$  the in Dirac basis (2.1):

$$\begin{aligned} G &= a_0 I + ib_0 \gamma^5 + iA_0 \gamma^0 + iA_k \gamma^k + B_0 \gamma^0 \gamma^5 + B_k \gamma^k \gamma^5 \\ &\quad + a_k 2\sigma_{0k} + b_1 2\sigma_{23} + b_2 2\sigma_{31} + b_3 2\sigma_{12} \\ &= a_0 I + ib_0 \gamma^5 + iA_0 \gamma^0 + A_k (i\gamma^k) + iB_0 (i\gamma^5 \gamma^0) - B_k (\gamma^5 \gamma^k) \\ &\quad + a_k (2\sigma_{0k}) - ib_1 (2i\sigma_{23}) - ib_2 (2i\sigma_{31}) - ib_3 (2i\sigma_{12}), \end{aligned}$$

with the help of the formulas

$$\begin{aligned} \gamma^0 + \gamma^5 \gamma^3 &= 2\lambda_4, & \gamma^0 - \gamma^5 \gamma^3 &= 2\lambda_{11}, & i\gamma^5 \gamma^0 + i\gamma^3 &= 2\lambda_5, \\ i\gamma^5 \gamma^0 - i\gamma^3 &= 2\lambda_{12}, & \gamma^5 \gamma^1 + i\gamma^2 &= 2\lambda_6, & \gamma^5 \gamma^1 - i\gamma^2 &= 2\lambda_9, \\ i\gamma^1 + \gamma^5 \gamma^2 &= 2\lambda_{10}, & i\gamma^1 - \gamma^5 \gamma^2 &= 2\lambda_7, & 2\sigma^{01} + 2i\sigma^{23} &= 2\lambda_1, \\ 2\sigma^{01} - 2i\sigma^{23} &= -2\lambda_{13}, & 2\sigma^{02} + 2i\sigma^{31} &= 2\lambda_2, & 2\sigma^{02} - 2i\sigma^{31} &= -2\lambda_{14}, \\ 2\sigma^{03} + 2i\sigma^{12} &= 2\lambda_3, & 2\sigma^{03} - 2i\sigma^{12} &= 2(\lambda'_{15} - \lambda'_8), & \gamma^5 + 2\sigma^{03} &= 2\lambda'_{15}, \\ \gamma^5 - 2\sigma^{03} &= 2(\lambda'_8 - \lambda_3), & \gamma^5 + 2i\sigma^{12} &= 2\lambda'_8, & \gamma^5 - 2i\sigma^{12} &= 2(\lambda'_{15} - \lambda_3) \end{aligned}$$

and inverse ones

$$\begin{aligned} \gamma^0 &= \lambda_4 + \lambda_{11}, & \gamma^5 \gamma^3 &= \lambda_4 - \lambda_{11}, & i\gamma^5 \gamma^0 &= \lambda_5 + \lambda_{12}, & i\gamma^3 &= \lambda_5 - \lambda_{12}, \\ \gamma^5 \gamma^1 &= \lambda_6 + \lambda_9, & i\gamma^2 &= \lambda_6 - \lambda_9, & i\gamma^1 &= \lambda_{10} + \lambda_7, & \gamma^5 \gamma^2 &= \lambda_{10} - \lambda_7, \\ 2i\sigma^{23} &= \lambda_1 + \lambda_{13}, & 2\sigma^{01} &= \lambda_1 - \lambda_{13}, & 2i\sigma^{31} &= \lambda_2 + \lambda_{14}, & 2\sigma^{02} &= \lambda_2 - \lambda_{14}, \\ 2\sigma^{03} &= \lambda_3 - (\lambda'_8 - \lambda'_{15}), & 2i\sigma^{12} &= \lambda_3 + (\lambda'_8 - \lambda'_{15}), & \gamma^5 &= -\lambda_3 + (\lambda'_8 + \lambda'_{15}), \end{aligned}$$

we will arrive at

$$\begin{aligned} G &= a_0 I + (a_1 - ib_1)\lambda_1 + (a_2 - ib_2)\lambda_2 + (iA_0 - B_3)\lambda_4 + (A_3 + iB_0)\lambda_5 + (A_2 - B_1)\lambda_6 \\ &\quad + (A_1 + B_2)\lambda_7 + (a_3 - ib_3 - ib_0)\lambda_3 + (-ib_3 + ib_0 - a_3)\lambda'_8 + (ib_0 + a_3 + ib_3)\lambda'_{15} \\ &\quad + (-B_1 - A_2)\lambda_9 + (A_1 - B_2)\lambda_{10} + (iA_0 + B_3)\lambda_{11} + (-A_3 + iB_0)\lambda_{12} \\ &\quad + (-a_1 - ib_1)\lambda_{13} + (-a_2 - ib_2)\lambda_{14}. \end{aligned}$$

In variables  $(k, m, l, n)$  (see (2.3), (2.4))

$$\begin{aligned} B_0 - iA_0 &= l_0, & B_j - iA_j &= l_j, & B_0 + iA_0 &= n_0, & B_j + iA_j &= n_j, \\ a_0 - ib_0 &= k_0, & a_j - ib_j &= k_j, & a_0 + ib_0 &= m_0, & a_j + ib_j &= m_j \end{aligned}$$

the previous expansion looks

$$\begin{aligned} G &= \frac{1}{2}(k_0 + m_0)I + k_1 \lambda_1 + k_2 \lambda_2 + \frac{1}{2}[(n_0 - n_3) - (l_0 + l_3)]\lambda_4 + \frac{1}{2i}[-(n_0 - n_3) - (l_0 + l_3)]\lambda_5 \\ &\quad + \frac{1}{2}[-(n_1 + in_2) - (l_1 - il_2)]\lambda_6 + \frac{1}{2i}[(n_1 + in_2) - (l_1 - il_2)]\lambda_7 + [k_3 + \frac{1}{2}(k_0 - m_0)]\lambda_3 \\ &\quad + [-m_3 + \frac{1}{2}(m_0 - k_0)]\lambda'_8 + [m_3 + \frac{1}{2}(m_0 - k_0)]\lambda'_{15} + \frac{1}{2}[-(n_1 - in_2) - (l_1 + il_2)]\lambda_9 \\ &\quad + \frac{1}{2i}[(n_1 - in_2) - (l_1 + il_2)]\lambda_{10} + \frac{1}{2}[(n_0 + n_3) - (l_0 - l_3)]\lambda_{11} \end{aligned}$$

$$+ \frac{1}{2i}[-(n_0 + n_3) - (l_0 - l_3)]\lambda_{12} - m_1\lambda_{13} - m_2\lambda_{14}.$$

Let coefficients at  $\lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}$  be equal to zero:

$$\begin{aligned} (n_1 - in_2) + (l_1 + il_2) &= 0, & (n_1 - in_2) - (l_1 + il_2) &= 0, & (n_0 + n_3) - (l_0 - l_3) &= 0, \\ (n_0 + n_3) + (l_0 - l_3) &= 0, & m_1 &= 0, & m_2 &= 0. \end{aligned} \quad (6.5)$$

Note that we do not require vanishing of the coefficient at  $\lambda'_{15}$ :

$$\lambda'_{15} \left( m_3 + \frac{m_0 - k_0}{2} \right) \neq 0.$$

As a result we have a subgroup of  $4 \times 4$  matrices defined by 10 complex parameters. At this four elements are diagonal matrices:

$$I, \quad \lambda_3, \quad \lambda'_8, \quad \lambda'_{15},$$

all other matrices have on the diagonal only zeros. Equations (6.5) give

$$in_2 = n_1, \quad n_3 = -n_0, \quad il_2 = -l_1, \quad l_3 = l_0, \quad m_1 = 0, \quad m_2 = 0, \quad (6.6)$$

so that any matrix  $G(k_a, n_0, n_1, l_0, l_1, m_0, m_3)$  is decomposed according to

$$\begin{aligned} G &= k_1\lambda_1 + k_2\lambda_2 + (n_0 - l_0)\lambda_4 + i(n_0 + l_0)\lambda_5 + (-n_1 - l_1)\lambda_6 + i(-n_1 + l_1)\lambda_7 + \frac{1}{2}(k_0 + m_0)I \\ &\quad + \left[ k_3 + \frac{1}{2}(k_0 - m_0) \right] \lambda_3 + \left[ -m_3 + \frac{1}{2}(m_0 - k_0) \right] \lambda'_8 + \left[ m_3 + \frac{1}{2}(m_0 - k_0) \right] \lambda'_{15}. \end{aligned} \quad (6.7)$$

Explicit form of the matrices parameterized by (6.7) can be obtained from representation for arbitrary element of  $GL(4, C)$  (2.6)

$$G(k, m, n, l) = \begin{vmatrix} +(k_0 + k_3) & +(k_1 - ik_2) & +(n_0 - n_3) & -(n_1 - in_2) \\ +(k_1 + ik_2) & +(k_0 - k_3) & -(n_1 + in_2) & +(n_0 + n_3) \\ -(l_0 + l_3) & -(l_1 - il_2) & +(m_0 - m_3) & -(m_1 - im_2) \\ -(l_1 + il_2) & -(l_0 - l_3) & -(m_1 + im_2) & +(m_0 + m_3) \end{vmatrix}$$

with additional restrictions (6.6):

$$G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & +2n_0 & 0 \\ k_1 + ik_2 & k_0 - k_3 & -2n_1 & 0 \\ -2l_0 & -2l_1 & m_0 - m_3 & 0 \\ 0 & 0 & 0 & m_0 + m_3 \end{vmatrix}. \quad (6.8)$$

If additionally one requires  $m_0 + m_3 = 1$ , then

$$G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & +2n_0 & 0 \\ k_1 + ik_2 & k_0 - k_3 & -2n_1 & 0 \\ -2l_0 & -2l_1 & 1 - 2m_3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

with decomposition rule

$$\begin{aligned} G &= k_1\lambda_1 + k_2\lambda_2 + (n_0 - l_0)\lambda_4 + i(n_0 + l_0)\lambda_5 + (-n_1 - l_1)\lambda_6 + i(-n_1 + l_1)\lambda_7 \\ &\quad + \frac{1}{2}(1 + k_0 - m_3)I + \left[ k_3 + \frac{1}{2}(k_0 + m_3 - 1) \right] \lambda_3 \\ &\quad + \left[ -m_3 + \frac{1}{2}(1 - m_3 - k_0) \right] \lambda'_8 + \frac{1}{2}(1 - k_0 + m_3)\lambda'_{15}. \end{aligned} \quad (6.9)$$

In the diagonal part of (6.9), there are four independent matrices because equation (6.9) represents  $4 \times 4$  matrix with the structure

$$G \sim \begin{vmatrix} GL(3, C) & 0 \\ 0 & 1 \end{vmatrix}.$$

To deal with the matrices from  $GL(3, C)$ , in the diagonal part of (6.9) one should separate only a  $3 \times 3$  block:

$$\begin{aligned} \text{Diag} &= \frac{1}{2}(1 + k_0 - m_3)I^{(3)} + [k_3 + \frac{1}{2}(k_0 + m_3 - 1)]\lambda_3^{(3)} \\ &\quad + [-m_3 + \frac{1}{2}(1 - m_3 - k_0)]\lambda_8'^{(3)} + \frac{1}{2}(1 - k_0 + m_3)\lambda_{15}'^{(3)} = \\ &= \frac{1}{2}(1 + k_0 - m_3) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + [k_3 + \frac{1}{2}(k_0 + m_3 - 1)] \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \\ &\quad + [-m_3 + \frac{1}{2}(1 - m_3 - k_0)] \begin{vmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \frac{1}{2}(1 - k_0 + m_3) \begin{vmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \end{aligned}$$

Resolving  $\lambda_{15}'^{(3)}$  in terms of  $I^{(3)}$ ,  $\lambda_3^{(3)}$ ,  $\lambda_8'^{(3)}$ :

$$\lambda_{15}'^{(3)} = -\frac{1}{3}I^{(3)} + \frac{1}{3}\lambda_3^{(3)} + \frac{1}{3}\lambda_8'^{(3)},$$

we arrive at a 3-term relation:

$$\text{Diag} = \frac{1}{3}(1 + 2k_0 - 2m_3)I^{(3)} + [k_3 + \frac{1}{3}(k_0 + 2m_3 - 1)]\lambda_3^{(3)} + \frac{1}{3}(-4m_3 - 2k_0 + 2)\lambda_8'^{(3)}.$$

The group law for parameters of  $SL(3, C)$  has the form (the notation  $M = 1 - 2m_3$  is used)

$$\begin{aligned} k_0'' &= k_0'k_0 + \mathbf{k}'\mathbf{k} + 2(-n_0'l_0 + n_1'l_1), \\ (\mathbf{k}'')_1 &= (k_0'\mathbf{k} + \mathbf{k}'k_0 + i\mathbf{k}' \times \mathbf{k})_1 + 2(-n_0'l_1 + n_1'l_0), \\ (\mathbf{k}'')_2 &= (k_0'\mathbf{k} + \mathbf{k}'k_0 + i\mathbf{k}' \times \mathbf{k})_2 + 2(-in_0'l_1 - in_1'l_0), \\ (\mathbf{k}'')_3 &= (k_0'\mathbf{k} + \mathbf{k}'k_0 + i\mathbf{k}' \times \mathbf{k})_3 + 2(-n_0'l_0 - n_1'l_1), \\ n_0'' &= (k_0' + k_3')n_0 - (k_1' - ik_2')n_1 + n_0'M, \\ n_1'' &= (k_0' - k_3')n_1 - (k_1' + ik_2')n_0 + n_1'M, \\ l_0'' &= l_0'(k_0 + k_3) + l_1'(k_1 + ik_2) + M'l_0, \\ l_1'' &= l_0'(k_1 - ik_2) + l_1'(k_0 - k_3) + M'l_1, \\ M'' &= M'M - 4(l_0'n_0 - l_1'n_1). \end{aligned}$$

These rules determine multiplication of the matrices

$$G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & +2n_0 & 0 \\ k_1 + ik_2 & k_0 - k_3 & -2n_1 & 0 \\ -2l_0 & -2l_1 & M & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

If additionally, in equation (6.8) one requires

$$n_0 = 0, \quad n_1 = 0, \quad l_0 = 0, \quad l_1 = 0, \quad m_3 = 0, \quad m_0 = 1,$$

then

$$G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & 0 & 0 \\ k_1 + ik_2 & +k_0 - k_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

with the decomposition rule

$$G = k_1\lambda_1 + k_2\lambda_2 + \frac{1}{2}(1 + k_0)I + [k_3 - \frac{1}{2}(1 - k_0)]\lambda_3 + \frac{1}{2}(1 - k_0)\lambda'_8 + \frac{1}{2}(1 - k_0)\lambda'_{15}.$$

One can readily verify that the  $2 \times 2$  block is given by

$$G^{(2)}(k_a) = k_0 I^{(2)} + k_1 \lambda_1^{(2)} + k_2 \lambda_2^{(2)} + k_3 \lambda_3^{(2)}.$$

## 7 On the multiplication law for $GL(4, C)$ in Dirac basis

In the Gell-Mann basis  $\lambda_i$ , an element of  $GL(4, C)$  is

$$\begin{aligned} G = & a_0 I + (a_1 - ib_1)\lambda_1 + (a_2 - ib_2)\lambda_2 + (iA_0 - B_3)\lambda_4 + (A_3 + iB_0)\lambda_5 + (A_2 - B_1)\lambda_6 \\ & + (A_1 + B_2)\lambda_7 + (a_3 - ib_3 - ib_0)\lambda_3 + (-ib_3 + ib_0 - a_3)\lambda'_8 + (-B_1 - A_2)\lambda_9 \\ & + (A_1 - B_2)\lambda_{10} + (iA_0 + B_3)\lambda_{11} + (-A_3 + iB_0)\lambda_{12} + (-a_1 - ib_1)\lambda_{13} \\ & + (-a_2 - ib_2)\lambda_{14} + (ib_0 + a_3 + ib_3)\lambda'_{15}, \end{aligned}$$

or in variables  $(k, m, l, n)$ :

$$\begin{aligned} G = & \frac{1}{2}(k_0 + m_0)I + k_1\lambda_1 + k_2\lambda_2 + \frac{1}{2}[(n_0 - n_3) - (l_0 + l_3)]\lambda_4 + \frac{1}{2i}[-(n_0 - n_3) - (l_0 + l_3)]\lambda_5 \\ & + \frac{1}{2}[-(n_1 + in_2) - (l_1 - il_2)]\lambda_6 + \frac{1}{2i}[(n_1 + in_2) - (l_1 - il_2)]\lambda_7 + [k_3 + \frac{1}{2}(k_0 - m_0)]\lambda_3 \\ & + [-m_3 + \frac{1}{2}(m_0 - k_0)]\lambda'_8 + \frac{1}{2}[-(n_1 - in_2) - (l_1 + il_2)]\lambda_9 \\ & + \frac{1}{2i}[(n_1 - in_2) - (l_1 + il_2)]\lambda_{10} + \frac{1}{2}[(n_0 + n_3) - (l_0 - l_3)]\lambda_{11} \\ & + \frac{1}{2i}[-(n_0 + n_3) - (l_0 - l_3)]\lambda_{12} - m_1\lambda_{13} - m_2\lambda_{14} + [m_3 + \frac{1}{2}(m_0 - k_0)]\lambda'_{15}. \end{aligned}$$

The problem is to establish the multiplication rule  $G'' = G'G$  in  $\lambda$ -basis:

$$x''_k \lambda_k = x'_m \lambda_m x_n \lambda_n = x'_m x_n \lambda_m \lambda_n.$$

As by definition the relationships  $\lambda_m \lambda_n = e_{mnk} \lambda_k$  must hold, the multiplication rule is

$$x''_k = e_{mnk} x'_m x_n. \quad (7.1)$$

The main claim is that the all properties of the  $GL(4, C)$  with all its subgroups are determined by the bilinear function (7.1), the latter is described by structure constants  $e_{mnk}$ . It is evident that these group constants should be simpler in the Dirac basis  $\Lambda_i$  than in the basis  $\lambda_i$ . Our next task is to establish the multiplication law  $G'' = G'G$  in  $\Lambda$ -basis:

$$\Lambda_m \Lambda_n = E_{mnk} \Lambda_k, \quad X''_k = E_{mnk} X'_m X_n.$$

Before searching for structural constants  $E_{mnk}$ , let us introduce a special way to list the Dirac basis  $\Lambda_i$ :

$$\begin{aligned} \alpha_1 &= \gamma^0 \gamma^2, & \alpha_2 &= i\gamma^0 \gamma^5, & \alpha_3 &= \gamma^5 \gamma^2 & \alpha_i^2 &= I, & \alpha_1 \alpha_2 &= i\alpha_3, \alpha_2 \alpha_1 = -i\alpha_1, \\ \beta_1 &= i\gamma^3 \gamma^1, & \beta_2 &= i\gamma^3, & \beta_3 &= i\gamma^1, & \beta_i^2 &= I, & \beta_1 \beta_2 &= i\beta_3, \beta_2 \beta_1 = -i\beta_3, \end{aligned}$$

these two set commute with each others  $\alpha_j \beta_k = \beta_k \alpha_j$ , and their multiplications provides us with 9 remaining basis elements of  $\{\Lambda_k\}$ :

$$\begin{aligned} A_1 &= \alpha_1 \beta_1 = -\gamma^5, & B_1 &= \alpha_1 \beta_2 = \gamma^5 \gamma^1, & C_1 &= \alpha_1 \beta_3 = \gamma^3 \gamma^5, \\ A_2 &= \alpha_2 \beta_1 = -i\gamma^2, & B_2 &= \alpha_2 \beta_2 = -i\gamma^1 \gamma^2, & C_2 &= \alpha_2 \beta_3 = -i\gamma^2 \gamma^3, \\ A_3 &= \alpha_3 \beta_1 = \gamma^0, & B_3 &= \alpha_3 \beta_2 = \gamma^0 \gamma^1, & C_3 &= \alpha_3 \beta_3 = \gamma^0 \gamma^3. \end{aligned} \quad (7.2)$$

The multiplication rules for basic elements

$$\alpha_1, \alpha_2, \alpha_3, \quad \beta_1, \beta_2, \beta_3, \quad A_1, A_2, A_3, \quad B_1, B_2, B_3, \quad C_1, C_2, C_3,$$

are

	$\alpha_1$	$\alpha_2$	$\alpha_3$		$\beta_1$	$\beta_2$	$\beta_3$		$A_1$	$A_2$	$A_3$
$\alpha_1$	$I$	$i\alpha_3$	$-i\alpha_2$	$\alpha_1$	$A_1$	$B_1$	$C_1$	$\alpha_1$	$\beta_1$	$iA_3$	$-iA_2$
$\alpha_2$	$-i\alpha_3$	$I$	$i\alpha_1$	$\alpha_2$	$A_2$	$B_2$	$C_2$	$\alpha_2$	$-iA_3$	$\beta_1$	$iA_1$
$\alpha_3$	$i\alpha_2$	$-i\alpha_1$	$I$	$\alpha_3$	$A_3$	$B_3$	$C_3$	$\alpha_3$	$iA_2$	$-iA_1$	$\beta_1$
	$B_1$	$B_2$	$B_3$		$C_1$	$C_2$	$C_3$		$\alpha_1$	$\alpha_2$	$\alpha_3$
$\alpha_1$	$\beta_2$	$iB_3$	$-iB_2$	$\alpha_1$	$\beta_3$	$iC_3$	$-iC_2$	$\beta_1$	$A_1$	$A_2$	$A_3$
$\alpha_2$	$-iB_3$	$\beta_2$	$iB_1$	$\alpha_2$	$-iC_3$	$\beta_3$	$iC_1$	$\beta_2$	$B_1$	$B_2$	$B_3$
$\alpha_3$	$iB_2$	$-iB_1$	$\beta_2$	$\alpha_3$	$iC_2$	$-iC_1$	$\beta_3$	$\beta_3$	$C_1$	$C_2$	$C_3$
	$\beta_1$	$\beta_2$	$\beta_3$		$A_1$	$A_2$	$A_3$		$B_1$	$B_2$	$B_3$
$\beta_1$	$I$	$i\beta_3$	$-i\beta_2$	$\beta_1$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\beta_1$	$iC_1$	$iC_2$	$iC_3$
$\beta_2$	$-i\beta_3$	$I$	$i\beta_1$	$\beta_2$	$-iC_1$	$-iC_2$	$-iC_3$	$\beta_2$	$\alpha_1$	$\alpha_2$	$\alpha_3$
$\beta_3$	$i\beta_2$	$-i\beta_1$	$I$	$\beta_3$	$iB_1$	$iB_2$	$iB_3$	$\beta_3$	$-iA_1$	$-iA_2$	$-iA_3$
	$C_1$	$C_2$	$C_3$		$\alpha_1$	$\alpha_2$	$\alpha_3$		$\beta_1$	$\beta_2$	$\beta_3$
$\beta_1$	$-iB_1$	$-iB_2$	$-iB_3$	$A_1$	$\beta_1$	$iA_3$	$-iA_2$	$A_1$	$\alpha_1$	$iC_1$	$-iB_1$
$\beta_2$	$iA_1$	$iA_2$	$iA_3$	$A_2$	$-iA_3$	$\beta_1$	$iA_1$	$A_2$	$\alpha_2$	$iC_2$	$-iB_2$
$\beta_3$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$A_3$	$iA_2$	$-iA_1$	$\beta_1$	$A_3$	$\alpha_3$	$iC_3$	$-iB_3$
	$A_1$	$A_2$	$A_3$		$B_1$	$B_2$	$B_3$		$C_1$	$C_2$	$C_3$
$A_1$	$I$	$i\alpha_3$	$-i\alpha_2$	$A_1$	$i\beta_3$	$-C_3$	$C_2$	$A_1$	$-i\beta_2$	$B_3$	$-B_2$
$A_2$	$-i\alpha_3$	$I$	$i\alpha_1$	$A_2$	$C_3$	$i\beta_3$	$-C_1$	$A_2$	$-B_3$	$-i\beta_2$	$B_1$
$A_3$	$i\alpha_2$	$-i\alpha_1$	$I$	$A_3$	$-C_2$	$C_1$	$i\beta_3$	$A_3$	$B_2$	$-B_1$	$-i\beta_2$
	$\alpha_1$	$\alpha_2$	$\alpha_3$		$\beta_1$	$\beta_2$	$\beta_3$		$A_1$	$A_2$	$A_3$
$B_1$	$\beta_2$	$iB_3$	$-iB_2$	$B_1$	$-iC_1$	$\alpha_1$	$iA_1$	$B_1$	$-i\beta_3$	$C_3$	$-C_2$
$B_2$	$-iB_3$	$\beta_2$	$iB_1$	$B_2$	$-iC_2$	$\alpha_2$	$iA_2$	$B_2$	$-C_3$	$-i\beta_3$	$C_1$
$B_3$	$iB_2$	$-iB_1$	$\beta_2$	$B_3$	$-iC_3$	$\alpha_3$	$iA_3$	$B_3$	$C_2$	$-C_1$	$-i\beta_3$
	$B_1$	$B_2$	$B_3$		$C_1$	$C_2$	$C_3$		$\alpha_1$	$\alpha_2$	$\alpha_3$
$B_1$	$I$	$i\alpha_3$	$-i\alpha_2$	$B_1$	$i\beta_1$	$-A_3$	$A_2$	$C_1$	$\beta_3$	$iC_3$	$-iC_2$
$B_2$	$-i\alpha_3$	$I$	$i\alpha_1$	$B_2$	$A_3$	$i\beta_1$	$-A_1$	$C_2$	$-iC_3$	$\beta_3$	$iC_1$
$B_3$	$i\alpha_2$	$-i\alpha_1$	$I$	$B_3$	$-A_2$	$A_1$	$i\beta_1$	$C_3$	$iC_2$	$-iC_1$	$\beta_3$
	$\beta_1$	$\beta_2$	$\beta_3$		$A_1$	$A_2$	$A_3$		$B_1$	$B_2$	$B_3$
$C_1$	$iB_1$	$-iA_1$	$\alpha_1$	$C_1$	$i\beta_2$	$-B_3$	$B_2$	$C_1$	$-i\beta_1$	$A_3$	$-A_2\beta_1$
$C_2$	$iB_2$	$-iA_2$	$\alpha_2$	$C_2$	$B_3$	$i\beta_2$	$-B_1$	$C_2$	$-A_3$	$-i\beta_1$	$A_1$
$C_3$	$iB_3$	$-iA_3$	$\alpha_3$	$C_3$	$-B_2$	$B_1$	$i\beta_2$	$C_3$	$A_2$	$-A_1$	$-i\beta_1$



$$\begin{array}{c|ccc}
& C_1 & C_2 & C_3 \\
\hline
C_1 & I & i\alpha_3 & -i\alpha_2 \\
C_2 & -i\alpha_3 & I & i\alpha_1 \\
C_3 & i\alpha_2 & -i\alpha_1 & I
\end{array} \tag{7.3}$$

These relations provide us with simple formulas for fifteen coordinates of the element of  $GL(4, C)$

$$\begin{aligned}
G &= \gamma I + a_j \alpha_j + b_j \beta_j + X_j A_j + Y_j B_j + Z_j C_j, \\
\gamma &= \frac{1}{4} \text{Sp } G, \quad a_j = \frac{1}{4} \text{Sp } \alpha_j G, \quad b_j = \frac{1}{4} \text{Sp } \beta_j G, \\
X_j &= \frac{1}{4} \text{Sp } A_j G, \quad Y_j = \frac{1}{4} \text{Sp } B_j G, \quad Z_j = \frac{1}{4} \text{Sp } C_j G.
\end{aligned}$$

With the use of relations (7.3) an explicit form of the group law for  $(15 + 1)$  parameters can be found:

$$\begin{aligned}
&(\gamma' I + a'_i \alpha_i + b'_i \beta_i + X'_i A_i + Y'_i B_i + Z'_i C_i)(\gamma I + a_j \alpha_j + b_j \beta_j + X_j A_j + Y_j B_j + Z_j C_j) \\
&= \gamma' \gamma I + (\gamma' a_j + a'_j \gamma) \alpha_j + (\gamma' b_j + b'_j \gamma) \beta_j + (\gamma' X_j + X'_j \gamma) A_j + (\gamma' Y_j + Y'_j \gamma) B_j \\
&+ (\gamma' Z_j + Z'_j \gamma) C_j + a'_1 (a_1 + a_2 i \alpha_3 - a_3 i \alpha_2) + a'_2 (-a_1 i \alpha_3 + a_2 + a_3 i \alpha_1) \\
&+ a'_3 (a_1 i \alpha_2 - a_2 i \alpha_1 + a_3) + a'_1 (b_1 A_1 + b_2 B_1 + b_3 C_1) + a'_2 (b_1 A_2 + b_2 B_2 + b_3 C_2) \\
&+ a'_3 (b_1 A_3 + b_2 B_3 + b_3 C_3) + a'_1 (X_1 \beta_1 + X_2 i A_3 - X_3 i A_2) \\
&+ a'_2 (-X_1 i A_3 + X_2 \beta_1 + X_3 i A_1) + a'_3 (X_1 i A_2 - X_2 i A_1 + X_3 \beta_1) \\
&+ a'_1 (Y_1 \beta_2 + Y_2 i B_3 - Y_3 i B_2) + a'_2 (-Y_1 i B_3 + Y_2 \beta_2 + Y_3 i B_1) \\
&+ a'_3 (Y_1 i B_2 - Y_2 i B_1 + Y_3 \beta_2) + a'_1 (Z_1 \beta_3 + Z_2 i C_3 - Z_3 i C_2) \\
&+ a'_2 (-Z_1 i C_3 + Z_2 \beta_3 + Z_3 i C_1) + a'_3 (Z_1 i C_2 - Z_2 i C_1 + Z_3 \beta_3) \\
&+ b'_1 (a_1 A_1 + a_2 A_2 + a_3 A_3) + b'_2 (a_1 B_1 + a_2 B_2 + a_3 B_3) + b'_3 (a_1 C_1 + a_2 C_2 + a_3 C_3) \\
&+ b'_1 (b_1 + b_2 i \beta_3 - b_3 i \beta_2) + b'_2 (-b_1 i \beta_3 + b_2 + b_3 i \beta_1) + b'_3 (b_1 i \beta_2 - b_2 i \beta_1 + b_3) \\
&+ b'_1 (X_1 \alpha_1 + X_2 \alpha_2 + X_3 \alpha_3) + b'_2 (-X_1 i C_1 - X_2 i C_2 - X_3 i C_3) \\
&+ b'_3 (X_1 i B_1 + X_2 i B_2 + X_3 i B_3) + b'_1 (Y_1 i C_1 + Y_2 i C_2 + Y_3 i C_3) \\
&+ b'_2 (Y_1 \alpha_1 + Y_2 \alpha_2 + Y_3 \alpha_3) + b'_3 (-Y_1 i A_1 - Y_2 i A_2 - Y_3 i A_3) \\
&+ b'_1 (-Z_1 i B_1 - Z_2 i B_2 - Z_3 i B_3) + b'_2 (Z_1 i A_1 + Z_2 i A_2 + Z_3 i A_3) \\
&+ b'_3 (Z_1 \alpha_1 + Z_2 \alpha_2 + Z_3 \alpha_3) + X'_1 (a_1 \beta_1 + a_2 i A_3 - a_3 i A_2) \\
&+ X'_2 (-a_1 i A_3 + a_2 \beta_1 + a_3 i A_1) + X'_3 (a_1 i A_2 - a_2 i A_1 + a_3 \beta_1) \\
&+ X'_1 (b_1 \alpha_1 + b_2 i C_1 - b_3 i B_1) + X'_2 (b_1 \alpha_2 + b_2 i C_2 - b_3 i B_2) + X'_3 (b_1 \alpha_3 + b_2 i C_3 - b_3 i B_3) \\
&+ X'_1 (X_1 + X_2 i \alpha_3 - X_3 i \alpha_2) + X'_2 (-X_1 i \alpha_3 + X_2 + X_3 i \alpha_1) + X'_3 (X_1 i \alpha_2 - X_2 i \alpha_1 + X_3) \\
&+ X'_1 (Y_1 i \beta_3 - Y_2 C_3 + Y_3 C_2) + X'_2 (Y_1 C_3 + Y_2 i \beta_3 - Y_3 C_1) + X'_3 (-Y_1 C_2 + Y_2 C_1 + Y_3 i \beta_3) \\
&+ X'_1 (-Z_1 i \beta_2 + Z_2 B_3 - Z_3 B_2) + X'_2 (-Z_1 B_3 - Z_2 i \beta_2 + Z_3 B_1) \\
&+ X'_3 (Z_1 B_2 - Z_2 B_1 - Z_3 i \beta_2) + Y'_1 (a_1 \beta_2 + a_2 i B_3 - a_3 i B_2) \\
&+ Y'_2 (-a_1 i B_3 + a_2 \beta_2 + a_3 i B_1) + Y'_3 (a_1 i B_2 - a_2 i B_1 + a_3 \beta_2) \\
&+ Y'_1 (-b_1 i C_1 + b_2 \alpha_1 + b_3 i A_1) + Y'_2 (-b_1 i C_2 + b_2 \alpha_2 + b_3 i A_2) \\
&+ Y'_3 (-b_1 i C_3 + b_2 \alpha_3 + b_3 i A_3) + Y'_1 (-X_1 i \beta_3 + X_2 C_3 - X_3 C_2) \\
&+ Y'_2 (-X_1 C_3 - X_2 i \beta_3 + X_3 C_1) + Y'_3 (X_1 C_2 - X_2 C_1 - X_3 i \beta_3) \\
&+ Y'_1 (Y_1 + Y_2 i \alpha_3 - Y_3 i \alpha_2) + Y'_2 (-Y_1 i \alpha_3 + Y_2 + Y_3 i \alpha_1) + Y'_3 (Y_1 i \alpha_2 - Y_2 i \alpha_1 + Y_3) \\
&+ Y'_1 (Z_1 i \beta_1 - Z_2 A_3 + Z_3 A_2) + Y'_2 (Z_1 A_3 + Z_2 i \beta_1 - Z_3 A_1) \\
&+ Y'_3 (-Z_1 A_2 + Z_2 A_1 + Z_3 i \beta_1) + Z'_1 (a_1 \beta_3 + a_2 i C_3 - a_3 i C_2) \\
&+ Z'_2 (-a_1 i C_3 + a_2 \beta_3 + a_3 i C_1) + Z'_3 (a_1 i C_2 - a_2 i C_1 + a_3 \beta_3)
\end{aligned}$$

$$\begin{aligned}
& + Z'_1(b_1iB_1 - b_2iA_1 + b_3\alpha_1) + Z'_2(b_1iB_2 - b_2iA_2 + b_3\alpha_2) + Z'_3(b_1iB_3 - b_2iA_3 + b_3\alpha_3) \\
& + Z'_1(X_1i\beta_2 - X_2B_3 + X_3B_2) + Z'_2(X_1B_3 + X_2i\beta_2 - X_3B_1) \\
& + Z'_3(-X_1B_2 + X_2B_1 + X_3i\beta_2) + Z'_1(-Y_1i\beta_1 + Y_2A_3 - Y_3A_2) \\
& + Z'_2(-Y_1A_3 - Y_2i\beta_1 + Y_3A_1) + Z'_3(Y_1A_2 - Y_2A_1 - Y_3i\beta_1) \\
& + Z'_1(Z_1 + Z_2i\alpha_3 - Z_3i\alpha_2) + Z'_2(-Z_1i\alpha_3 + Z_2 + Z_3i\alpha_1) + Z'_3(Z_1i\alpha_2 - Z_2i\alpha_1 + Z_3).
\end{aligned}$$

From these relations we arrive at the following composition rules:

$$\begin{aligned}
\gamma'' &= \gamma'\gamma + (a'_1a_1 + a'_2a_2 + a'_3a_3) + (b'_1b_1 + b'_2b_2 + b'_3b_3) \\
&+ (X'_1X_1 + X'_2X_2 + X'_3X_3) + (Y'_1Y_1 + Y'_2Y_2 + Y'_3Y_3) + (Z'_1Z_1 + Z'_2Z_2 + Z'_3Z_3), \\
a''_1 &= (\gamma'a_1 + a'_1\gamma) + (b'_1X_1 + b'_2Y_1 + b'_3Z_1) + (X'_1b_1 + Y'_1b_2 + Z'_1b_3) \\
&+ i(a'_2a_3 - a'_3a_2) + i(X'_2X_3 - X'_3X_2) + i(Y'_2Y_3 - Y'_3Y_2) + i(Z'_2Z_3 - Z'_3Z_2), \\
a''_2 &= (\gamma'a_2 + a'_2\gamma) + (b'_1X_2 + b'_2Y_2 + b'_3Z_2) + (X'_2b_1 + Y'_2b_2 + Z'_2b_3) \\
&+ i(a'_3a_1 - a'_1a_3) + i(X'_3X_1 - X'_1X_3) + i(Y'_3Y_1 - Y'_1Y_3) + i(Z'_3Z_1 - Z'_1Z_3), \\
a''_3 &= (\gamma'a_3 + a'_3\gamma) + (b'_1X_3 + b'_2Y_3 + b'_3Z_3) + (X'_3b_1 + Y'_3b_2 + Z'_3b_3) \\
&+ i(a'_1a_2 - a'_2a_1) + i(X'_1X_2 - X'_2X_1) + i(Y'_1Y_2 - Y'_2Y_1) + i(Z'_1Z_2 - Z'_2Z_1), \\
b''_1 &= \gamma'b_1 + b'_1\gamma + i(b'_2b_3 - b'_3b_2) + (a'_1X_1 + a'_2X_2 + a'_3X_3) + (X'_1a_1 + X'_2a_2 + X'_3a_3) \\
&+ i(Y'_1Z_1 + Y'_2Z_2 + Y'_3Z_3) - i(Z'_1Y_1 + Z'_2Y_2 + Z'_3Y_3), \\
b''_2 &= \gamma'b_2 + b'_2\gamma + i(b'_3b_1 - b'_1b_3) + (a'_1Y_1 + a'_2Y_2 + a'_3Y_3) + (Y'_1a_1 + Y'_2a_2 + Y'_3a_3) \\
&+ i(Z'_1X_1 + Z'_2X_2 + Z'_3X_3) - i(X'_1Z_1 + X'_2Z_2 + X'_3Z_3), \\
b''_3 &= \gamma'b_3 + b'_3\gamma + i(b'_1b_2 - b'_2b_1) + (a'_1Z_1 + a'_2Z_2 + a'_3Z_3) + (Z'_1a_1 + Z'_2a_2 + Z'_3a_3) \\
&+ i(X'_1Y_1 + X'_2Y_2 + X'_3Y_3) - i(Y'_1X_1 + Y'_2X_2 + Y'_3X_3), \\
X''_1 &= (\gamma'X_1 + \gamma X'_1) + (a'_1b_1 + a_1b'_1) + i(Y'_1b_3 - Y_1b'_3) + i(b'_2Z_1 - b_2Z'_1) \\
&+ i(a'_2X_3 - a'_3X_2) - i(a_2X'_3 - a_3X'_2) + (Z_2Y'_3 - Z_3Y'_2) + (Z'_2Y_3 - Z'_3Y_2), \\
X''_2 &= (\gamma'X_2 + \gamma X'_2) + (a'_2b_1 + a_2b'_1) + i(Y'_2b_3 - Y_2b'_3) + i(b'_2Z_2 - b_2Z'_2) \\
&+ i(a'_3X_1 - a'_1X_2) - i(a_3X'_1 - a_1X'_2) + (Z_3Y'_1 - Z_1Y'_3) + (Z'_3Y_1 - Z'_1Y_3), \\
X''_3 &= (\gamma'X_3 + \gamma X'_3) + (a'_3b_1 + a_3b'_1) + i(Y'_3b_3 - Y_3b'_3) + i(b'_2Z_3 - b_2Z'_3) \\
&+ i(a'_1X_2 - a'_2X_1) - i(a_1X'_2 - a_2X'_1) + (Z_1Y'_2 - Z_2Y'_1) + (Z'_1Y_2 - Z'_2Y_1), \\
Y''_1 &= (\gamma'Y_1 + \gamma Y'_1) + (a'_1b_2 + a_1b'_2) + i(Z'_1b_1 - Z_1b'_1) + i(b'_3X_1 - b_3X'_1) \\
&+ i(a'_2Y_3 - a'_3Y_2) - i(a_2Y'_3 - a_3Y'_2) + (X_2Z'_3 - X_3Z'_2) + (X'_2Z_3 - X'_3Z_2), \\
Y''_2 &= (\gamma'Y_2 + \gamma Y'_2) + (a'_2b_2 + a_2b'_2) + i(Z'_2b_1 - Z_2b'_1) + i(b'_3X_2 - b_3X'_2) \\
&+ i(a'_3Y_1 - a'_1Y_3) - i(a_3Y'_1 - a_1Y'_3) + (X_3Z'_1 - X_1Z'_3) + (X'_3Z_1 - X'_1Z_3), \\
Y''_3 &= (\gamma'Y_3 + \gamma Y'_3) + (a'_3b_2 + a_3b'_2) + i(Z'_3b_1 - Z_3b'_1) + i(b'_3X_3 - b_3X'_3) \\
&+ i(a'_1Y_2 - a'_2Y_1) - i(a_1Y'_2 - a_2Y'_1) + (X_1Z'_2 - X_2Z'_1) + (X'_1Z_2 - X'_2Z_1), \\
Z''_1 &= (\gamma'Z_1 + \gamma Z'_1) + (a'_1b_3 + a_1b'_3) + i(Y_1b'_1 - Y'_1b_1) + i(X'_1b_2 - X_1b'_2) \\
&+ i(a'_2Z_3 - a'_3Z_2) - i(a_2Z'_3 - a_3Z'_2) + (Y_2X'_3 - Y_3X'_2) + (Y'_2X_3 - Y'_3X_2), \\
Z''_2 &= (\gamma'Z_2 + \gamma Z'_2) + (a'_2b_3 + a_2b'_3) + i(Y_2b'_1 - Y'_2b_1) + i(X'_2b_2 - X_2b'_2) \\
&+ i(a'_3Z_1 - a'_1Z_3) - i(a_3Z'_1 - a_1Z'_3) + (Y_3X'_1 - Y_1X'_3) + (Y'_3X_1 - Y'_1X_3), \\
Z''_3 &= (\gamma'Z_3 + \gamma Z'_3) + (a'_3b_3 + a_3b'_3) + i(Y_3b'_1 - Y'_3b_1) + i(X'_3b_2 - X_3b'_2) \\
&+ i(a'_1Z_2 - a'_2Z_1) - i(a_1Z'_2 - a_2Z'_1) + (Y_1X'_2 - Y_2X'_1) + (Y'_1X_2 - Y'_2X_1). \tag{7.4}
\end{aligned}$$

With the help of the index notation

$$\mathbf{X} = \mathbf{C}^{(1)}, \quad \mathbf{Y} = \mathbf{C}^{(2)}, \quad \mathbf{Z} = \mathbf{C}^{(3)},$$

it is easy to see a cyclic symmetry in the above relationships:

$$\begin{aligned}
\gamma'' &= \gamma' \gamma + a'_k a_k + b'_k b_k + C_k^{(1)'} C_k^{(1)} + C_k^{(2)'} C_k^{(2)} + C_k^{(3)'} C_k^{(3)}, \\
a''_k &= \gamma' a_k + \gamma a'_k + (b'_1 C_k^{(1)} + b_1 C_k^{(1)'}) + (b'_2 C_k^{(2)} + b_2 C_k^{(2)'}) + (b'_3 C_k^{(3)} + b_3 C_k^{(3)'}) \\
&\quad + i\epsilon_{kln} a'_l a_n + i\epsilon_{kln} C_l^{(1)'} C_n^{(1)} + i\epsilon_{kln} C_l^{(2)'} C_n^{(2)} + i\epsilon_{kln} C_l^{(3)'} C_n^{(3)}, \\
b''_k &= \gamma' b_k + \gamma b'_k + i\epsilon_{kln} b'_l b_n + (a'_1 C_1^{(k)} + a_1 C_1^{(k)'}) + (a'_2 C_2^{(k)} + a_2 C_2^{(k)'}) \\
&\quad + (a'_3 C_3^{(k)} + a_3 C_3^{(k)'}) + i\epsilon_{kln} C_m^{(l)'} C_m^{(n)}, \\
C_k^{(1)} &= \gamma' C_k^{(1)} + \gamma C_k^{(1)'} + (a'_k b_1 + a_k b'_1) + i\epsilon_{(1)ln} (C_k^{(l)'} b_n - C_k^{(l)} b'_n) \\
&\quad + i\epsilon_{kln} (a'_l C_n^{(1)} - a_l C_n^{(1)'}) + \epsilon_{kln} (C_l^{(2)'} C_n^{(3)} + C_l^{(2)} C_n^{(3)'}), \\
C_k^{(2)} &= \gamma' C_k^{(2)} + \gamma C_k^{(2)'} + (a'_k b_2 + a_k b'_2) + i\epsilon_{(2)ln} (C_k^{(l)'} b_n - C_k^{(l)} b'_n) \\
&\quad + i\epsilon_{kln} (a'_l C_n^{(2)} - a_l C_n^{(2)'}) + \epsilon_{kln} (C_l^{(3)'} C_n^{(1)} + C_l^{(3)} C_n^{(1)'}), \\
C_k^{(3)} &= \gamma' C_k^{(3)} + \gamma C_k^{(3)'} + (a'_k b_3 + a_k b'_3) + i\epsilon_{(3)ln} (C_k^{(l)'} b_n - C_k^{(l)} b'_n) \\
&\quad + i\epsilon_{kln} (a'_l C_n^{(3)} - a_l C_n^{(3)'}) + \epsilon_{kln} (C_l^{(1)'} C_n^{(2)} + C_l^{(1)} C_n^{(2)'}).
\end{aligned} \tag{7.5}$$

It is readily seen that these group multiplication laws (7.4), (7.5) permit 15 two-parametric subgroups:

$$\begin{aligned}
(\gamma, a) \in \{ & (\gamma, a_1), (\gamma, a_2), (\gamma, a_3), (\gamma, b_1), (\gamma, b_2), (\gamma, b_3), (\gamma, X_1), (\gamma, X_2), (\gamma, X_3), \\
& (\gamma, Y_1), (\gamma, Y_2), (\gamma, Y_3), (\gamma, Z_1), (\gamma, Z_2), (\gamma, Z_3) \}
\end{aligned}$$

with the same composition law:

$$\gamma'' = \gamma' \gamma + a' a, \quad a'' = \gamma' a + \gamma a',$$

which in variables  $\gamma = W \cos \phi$ ,  $a = iW \sin \phi$  takes the form

$$W'' = W' W, \quad \alpha'' = \alpha' + \alpha.$$

The variable  $W$  is determined by  $\det G(W, \alpha) = W^4$ , the choice  $W = 1$  guarantees  $\det G = +1$ .

All 15 basis elements  $\Lambda_{(\rho)} \in \{\alpha_k, \beta_k, A_k, B_k, C_k\}$  possess the same properties:

$$\Lambda_{(\rho)}^+ = \Lambda_{(\rho)}, \quad \Lambda_{(\rho)}^2 = I.$$

Therefore, one can construct 15 different elementary unitary (at real valued parameters) matrices by one the same recipe:

$$\begin{aligned}
U_{(\rho)} &= e^{i\phi_{(\rho)} \Lambda_{(\rho)}} = \cos \phi_{(\rho)} + i \sin \phi_{(\rho)} \Lambda_{(\rho)}, \\
U_{(\rho)}^+ &= U_{(\rho)}^{-1} = e^{-i\phi_{(\rho)} \Lambda_{(\rho)}} = \cos \phi_{(\rho)} - i \sin \phi_{(\rho)} \Lambda_{(\rho)}.
\end{aligned}$$

The whole set of unitary matrices  $SU(4)$  may be constructed on the basis of a simple factorized formula:

$$U = e^{i\phi_{(1)} \Lambda_{(1)}} \dots e^{i\phi_{(15)} \Lambda_{(15)}}.$$

The order of the factors is important. Every such order leads us to a definite parametrization for the group  $SU(4)$  – all them seem to be equivalent.

In the end of the section let us write down the explicit form of these 15 elementary unitary transformations:

$$\begin{aligned}
\alpha_1 &= \begin{vmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{vmatrix}, & \alpha_2 &= \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix}, & \alpha_3 &= \begin{vmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{vmatrix}, \\
U_1^\alpha &= \begin{vmatrix} \cos \phi + i \sin \phi \sigma_2 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_2 \end{vmatrix}, & U_2^\alpha &= \begin{vmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{vmatrix}, \\
U_3^\alpha &= \begin{vmatrix} \cos \phi & i \sin \phi \sigma_2 \\ i \sin \phi \sigma_2 & \cos \phi \end{vmatrix}, & \beta_1 &= \begin{vmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{vmatrix}, & \beta_2 &= \begin{vmatrix} 0 & -i \sigma_3 \\ i \sigma_3 & 0 \end{vmatrix}, \\
\beta_3 &= \begin{vmatrix} 0 & -i \sigma_1 \\ i \sigma_1 & 0 \end{vmatrix}, & U_1^\beta &= \begin{vmatrix} \cos \phi + i \sin \phi \sigma_2 & 0 \\ 0 & \cos \phi + i \sin \phi \sigma_2 \end{vmatrix}, \\
U_2^\beta &= \begin{vmatrix} \cos \phi & \sin \phi \sigma_3 \\ -\sin \phi \sigma_3 & \cos \phi \end{vmatrix}, & U_3^\beta &= \begin{vmatrix} \cos \phi & \sin \phi \sigma_1 \\ -\sin \phi \sigma_1 & \cos \phi \end{vmatrix}, & A_1 &= \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \\
A_2 &= \begin{vmatrix} 0 & i \sigma_2 \\ -i \sigma_2 & 0 \end{vmatrix}, & A_3 &= \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, & U_1^A &= \begin{vmatrix} \cos \phi + i \sin \phi & 0 \\ 0 & \cos \phi - i \sin \phi \end{vmatrix}, \\
U_2^A &= \begin{vmatrix} \cos \phi & -\sin \phi \sigma_2 \\ \sin \phi \sigma_2 & \cos \phi \end{vmatrix}, & U_3^A &= \begin{vmatrix} \cos \phi & i \sin \phi \\ i \sin \phi & \cos \phi \end{vmatrix}, & B_1 &= \begin{vmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{vmatrix}, \\
B_2 &= \begin{vmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{vmatrix}, & B_3 &= \begin{vmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, & U_1^B &= \begin{vmatrix} \cos \phi & i \sin \phi \sigma_1 \\ i \sin \phi \sigma_1 & \cos \phi \end{vmatrix}, \\
U_2^B &= \begin{vmatrix} \cos \phi - i \sin \phi \sigma_3 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_3 \end{vmatrix}, \\
U_3^B &= \begin{vmatrix} \cos \phi - i \sin \phi \sigma_1 & 0 \\ 0 & \cos \phi + i \sin \phi \sigma_1 \end{vmatrix}, & C_1 &= \begin{vmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{vmatrix}, \\
C_2 &= \begin{vmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{vmatrix}, & C_3 &= \begin{vmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{vmatrix}, & U_1^C &= \begin{vmatrix} \cos \phi & -i \sin \phi \sigma_3 \\ -i \sin \phi \sigma_3 & \cos \phi \end{vmatrix}, \\
U_2^C &= \begin{vmatrix} \cos \phi - i \sin \phi \sigma_1 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_1 \end{vmatrix}, \\
U_3^C &= \begin{vmatrix} \cos \phi + i \sin \phi \sigma_3 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_3 \end{vmatrix}. \tag{7.6}
\end{aligned}$$

Certainly, these relations provide us with 15 elementary solutions of the unitarity equations (3.3). For instance, the generator  $\alpha_2$  gives rise to the above 1-parametric Abelian subgroup  $G^0(\alpha)$ ; whereas the above 4-parametric subgroup  $G_0 \times SU(2)$  (5.15) is generated by  $(\alpha_2; \beta_1, B_2, C_2)$ .

The question is how one could describe all combinations of the above 15 simple sub-solutions by a single unifying formula – the latter should evidently exist.

## 8 On factorization $SU(4)$ and the group fine-structure

On the basis of 9 matrices (7.2) one can construct six 3-dimensional sub-sets:

$$\begin{aligned}
\mathbf{K} &= \{A_1 = \alpha_1 \beta_1, B_2 = \alpha_2 \beta_2, C_3 = \alpha_3 \beta_3\}, \\
\mathbf{L} &= \{C_1 = \alpha_1 \beta_3, A_2 = \alpha_2 \beta_1, B_3 = \alpha_3 \beta_2\}, \\
\mathbf{M} &= \{B_1 = \alpha_1 \beta_2, C_2 = \alpha_2 \beta_3, A_3 = \alpha_3 \beta_1\}, \\
\mathbf{K}' &= \{-C'_1 = -\alpha_1 \beta_3, -B'_2 = -\alpha_2 \beta_2, -C'_3 = -\alpha_3 \beta_3\}, \\
\mathbf{L}' &= \{-B'_1 = -\alpha_1 \beta_2, -A'_2 = -\alpha_2 \beta_1, -B'_3 = -\alpha_3 \beta_2\}, \\
\mathbf{M}' &= \{-A'_1 = -\alpha_1 \beta_1, -C'_2 = -\alpha_2 \beta_3, -B'_3 = -\alpha_3 \beta_2\},
\end{aligned}$$

(one may recall the rule to calculate the determinant of a  $3 \times 3$  matrix) with the same commutation relations:

$$\Gamma_1 \Gamma_2 = -\Gamma_3, \quad \Gamma_2 \Gamma_1 = -\Gamma_3, \quad \Gamma_1 \Gamma_2 - \Gamma_2 \Gamma_1 = 0, \quad \Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_1 = -2\Gamma_3, \quad (8.1)$$

and analogous by cyclic symmetry. The whole set of the above 9 matrices coincides with

$$\vec{\alpha}, \quad \vec{\beta}, \quad \mathbf{K}, \quad \mathbf{L}, \quad \mathbf{M}, \quad (8.2)$$

or

$$\vec{\alpha}, \quad \vec{\beta}, \quad \mathbf{K}', \quad \mathbf{L}', \quad \mathbf{M}'.$$

It suffices to consider one variant, let it be (8.2). It seem reasonable to suppose that arbitrary element from  $GL(4, C)$  can be factorized as follows

$$S = e^{i\vec{\alpha}\vec{\alpha}} e^{i\vec{\beta}\vec{\beta}} e^{i\mathbf{K}\mathbf{K}} e^{i\mathbf{L}\mathbf{L}} e^{i\mathbf{M}\mathbf{M}}. \quad (8.3)$$

When all parameters are real-valued, the formula provides us with the rule to construct elements from  $SU(4)$  group<sup>14</sup>. The order of factors might be different. Let us specify the group law for these 5 subsets. First are the two groups:

$$e^{i\vec{\alpha}\vec{\alpha}} = \cos a + i \sin a (n_1 \alpha_1 + n_2 \alpha_2 + n_3 \alpha_3), \quad e^{i\vec{\beta}\vec{\beta}} = \cos b + i \sin b (n_1 \beta_1 + n_2 \beta_2 + n_3 \beta_3).$$

They are isomorphic, so one can consider only the first one:

$$e^{i\vec{\alpha}\vec{\alpha}} = \cos a + i \sin a (n_1 \alpha_1 + n_2 \alpha_2 + n_3 \alpha_3) = x_0 - ix_1 \alpha_1 - ix_2 \alpha_2 - ix_3 \alpha_3.$$

Multiplying two matrices we arrive at

$$\begin{aligned} x_0'' &= x_0' x_0 - x_1' x_1 - x_2' x_2 - x_3' x_3, & x_1'' &= x_0' x_1 + x_1' x_0 + (x_2' x_3 - x_2' x_3), \\ x_2'' &= x_0' x_2 + x_2' x_0 + (x_3' x_1 - x_1' x_3), & x_3'' &= x_0' x_3 + x_3' x_0 + (x_1' x_2 - x_2' x_1). \end{aligned} \quad (8.4)$$

Parameters  $(x_0, x_i)$  should obey

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \quad \Longleftrightarrow \quad \det e^{i\vec{\alpha}\vec{\alpha}} = +1.$$

The inverse matrix looks

$$(x_0, \mathbf{x})^{-1} = (x_0, -\mathbf{x}).$$

With real  $(x_0, x_i)$  we have a group isomorphic to  $SU(2)$ , spinor covering for  $SO(3, R)$ :

$$\mathbf{c} = \frac{\mathbf{x}}{x_0}, \quad \mathbf{c}'' = \frac{\mathbf{c}' + \mathbf{c} + \mathbf{c}' \times \mathbf{c}}{1 - \mathbf{c}' \mathbf{c}}.$$

At complex  $(x_0, x_i)$  we have a group isomorphic to  $GL(2, C)$ , spinor covering for  $SO(3, C)$  or Lorentz group.

Now let us turn to finite transformations from remaining subsets. It is readily verified that these 1-parametric finite elements

$$e^{iy_1 \Gamma_1} = \cos y_1 + i \sin y_1 \Gamma_1, \quad e^{iy_2 \Gamma_1} = \cos y_2 + i \sin y_2 \Gamma_2, \quad e^{iy_3 \Gamma_3} = \cos y_3 + i \sin y_3 \Gamma_3,$$

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<sup>14</sup>Just such a structure was described in [40].

commute with each other:

$$\begin{aligned}
 e^{iy_1\Gamma_1}e^{iy_2\Gamma_2} &= (\cos y_1 + i \sin y_1\Gamma_1)(\cos y_2 + i \sin y_2\Gamma_2) = \\
 &= \cos y_1 \cos y_2 + i \cos y_1 \sin y_2\Gamma_2 + i \cos y_2 \sin y_1\Gamma_1 + \sin y_1 \sin y_2\Gamma_3, \\
 e^{iy_2\Gamma_2}e^{iy_1\Gamma_1} &= (\cos y_2 + i \sin y_2\Gamma_2)(\cos y_1 + i \sin y_1\Gamma_1) = \\
 &= \cos y_2 \cos y_1 + i \cos y_2 \sin y_1\Gamma_1 + i \cos y_1 \sin y_2\Gamma_2 + \sin y_2 \sin y_1\Gamma_3,
 \end{aligned}$$

that is  $e^{iy_1\Gamma_1}e^{iy_2\Gamma_2} = e^{iy_2\Gamma_2}e^{iy_1\Gamma_1}$ , and so on. Evidently, this property correlates with the commutative relations (8.1). Thus, each of tree subgroups can be constructed as multiplying of elementary 1-parametric commuting transformations. Their explicit forms are:

subgroup  $K$

$$\begin{aligned}
 \mathbf{K} &= \{A_1 = \alpha_1\beta_1, B_2 = \alpha_2\beta_2, C_3 = \alpha_3\beta_3\}, \\
 A_1 &= \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \quad B_2 = \begin{vmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{vmatrix}, \quad C_3 = \begin{vmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{vmatrix}, \\
 e^{ik_1K_1} &= \cos k_1 + i \sin k_1 A_1, \quad e^{ik_2K_2} = \cos k_2 + i \sin k_2 B_2, \\
 e^{ik_3K_3} &= \cos k_3 + i \sin k_3 C_3;
 \end{aligned}$$

subgroup  $L$

$$\begin{aligned}
 \mathbf{L} &= \{C_1 = \alpha_1\beta_3, A_2 = \alpha_2\beta_1, B_3 = \alpha_3\beta_2\}, \\
 C_1 &= \begin{vmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{vmatrix}, \quad A_2 = \begin{vmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{vmatrix}, \quad B_3 = \begin{vmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, \\
 e^{il_1L_1} &= \cos l_1 + i \sin l_1 C_1, \quad e^{il_2L_2} = \cos l_2 + i \sin l_2 A_2, \quad e^{il_3L_3} = \cos l_3 + i \sin l_3 B_3;
 \end{aligned}$$

subgroup  $M$

$$\begin{aligned}
 \mathbf{M} &= \{B_1 = \alpha_1\beta_2, C_2 = \alpha_2\beta_3, A_3 = \alpha_3\beta_1\}, \\
 B_1 &= \begin{vmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{vmatrix}, \quad C_2 = \begin{vmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{vmatrix}, \quad A_3 = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \\
 e^{im_1M_1} &= \cos m_1 + i \sin m_1 B_1, \quad e^{im_2M_2} = \cos m_2 + i \sin m_2 C_2, \\
 e^{im_3M_3} &= \cos m_3 + i \sin m_3 A_3.
 \end{aligned}$$

One additional note should be made. In the recent paper by A. Gsponer [35] on the quaternion approach to the problem of building the finite transformations from  $SU(3)$  and  $SU(4)$  an important point was to divide 15 basis  $4 \times 4$  matrices into three sets:

- set  $A$  of antisymmetrical matrices,
- set  $S$  of symmetrical matrices,
- set  $D$  of diagonal traceless ones.

It is easily seen that

- set  $A = \{\alpha_i \oplus \beta_i\}$  ;
- set  $S = \{A_2, A_3, B_1, B_3, C_1, C_2\} = \{\mathbf{L} \oplus \mathbf{M}\}$ ;
- set  $D = \{A_1, B_2, C_3 = \mathbf{K}\}$ .

Turning again to relationship (8.3), let us rewrite it as follows

$$S = e^{i\vec{a}\vec{\alpha}} [e^{i\vec{k}\vec{K}} e^{i\vec{l}\vec{L}} e^{i\vec{m}\vec{M}}] e^{i\vec{b}\vec{\beta}},$$

which exactly corresponds to the structure used in [35] in connection with the Lanczos decomposition theorem [43].

Several last comments should be made. On the basis of 15 matrices

$$\begin{aligned} \alpha_1, & \quad \alpha_2, & \quad \alpha_3, & \quad \beta_1, & \quad \beta_2, & \quad \beta_3, \\ A_1 = \alpha_1\beta_1, & \quad B_1 = \alpha_1\beta_2, & \quad C_1 = \alpha_1\beta_3, \\ A_2 = \alpha_2\beta_1, & \quad B_2 = \alpha_2\beta_2, & \quad C_2 = \alpha_2\beta_3, \\ A_3 = \alpha_3\beta_1, & \quad B_3 = \alpha_3\beta_2, & \quad C_3 = \alpha_3\beta_3 \end{aligned}$$

one can easily see the following 20 ways to separate  $SU(2)$  subgroups (certainly, those arise at real-valued parameters; complex-valued parameters give rise to linear subgroups  $GL(2, C)$ ):

$$\begin{aligned} (\alpha_1, \alpha_2, \alpha_3), & \quad (\beta_1, \beta_2, \beta_3), \\ (\alpha_1, A_2, A_3), & \quad (A_1, \alpha_2, A_3), & \quad (A_1, A_2, \alpha_3), \\ (\alpha_1, B_2, B_3), & \quad (B_1, \alpha_2, B_3), & \quad (B_1, B_2, \alpha_3), \\ (\alpha_1, C_2, C_3), & \quad (C_1, \alpha_2, C_3), & \quad (C_1, C_2, \alpha_3), \\ (\beta_1, B_1, C_1), & \quad (\beta_1, B_2, C_2), & \quad (\beta_1, B_3, C_3), \\ (A_1, \beta_2, C_1), & \quad (A_2, \beta_2, C_2), & \quad (A_3, \beta_2, C_3), \\ (A_1, B_1, \beta_3), & \quad (A_2, B_2, \beta_3), & \quad (A_3, B_3, \beta_3). \end{aligned} \tag{8.5}$$

Certainly, they provide us with twenty different 3-parametric solutions of the unitarity equations (3.3). Such 3-subgroups might be used as bigger elementary blocks in constructing of a general transformation [25, 28].

For instance, for the variant from (8.5):  $(\alpha_1, A_2, A_3) \implies (a_1, X_2, X_3)$  the general multiplication law (7.4) gives

$$\begin{aligned} \gamma'' &= \gamma'\gamma + a'_1a_1 + X'_2X_2 + X'_3X_3, & a_1'' &= \gamma'a_1 + a'_1\gamma + i(X'_2X_3 - X'_3X_2), \\ a_2'' &= 0, & a_3'' &= 0, & b_1'' &= 0, & b_2'' &= 0, & b_3'' &= 0, & X_1'' &= 0, \\ X_2'' &= \gamma'X_2 + \gamma X'_2 + i(-a'_1X_2) + a_1X'_3, & X_3'' &= \gamma'X_3 + \gamma X'_3 + i(a'_1X_2 - a_1X'_2), \\ Y_1'' &= 0, & Y_2'' &= 0, & Y_3'' &= 0, & Z_1'' &= 0, & Z_2'' &= 0, & Z_3'' &= 0, \end{aligned}$$

that is

$$\begin{aligned} \gamma'' &= \gamma'\gamma + a'_1a_1 + X'_2X_2 + X'_3X_3, & a_1'' &= \gamma'a_1 + a'_1\gamma + i(X'_2X_3 - X'_3X_2), \\ X_2'' &= \gamma'X_2 + \gamma X'_2 + i(X'_3a_1 - a'_1X_2), & X_3'' &= \gamma'X_3 + \gamma X'_3 + i(a'_1X_2 - a_1X'_2), \end{aligned}$$

which coincides with equation (8.4). The same can be done for any other representative from (8.5).

## 9 On pseudo-unitary group $SU(2, 2)$

As said, the Dirac basis was used previously [9, 40, 48, 50] in studying the exponentials for  $SU(2, 2)$  matrices. Let us show how the above formalism can apply to this pseudo-unitary group  $SU(2, 2)$ . Transformations from  $SU(2, 2)$  should leave invariant the following form

$$(z_1^*, z_2^*, z_3^*, z_4^*) \begin{vmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{vmatrix}, \quad z^+\eta z = z'^+\eta z',$$



which leads to

$$z' = Uz, \quad \text{where} \quad U^+ \eta = \eta U^{-1}, \quad \eta = \begin{vmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}.$$

Any generator  $\Lambda'$  of those transformations must obey relation

$$U_k = e^{ia\Lambda'_k}, \quad (\Lambda'_k)^+ \eta = \eta \Lambda'_k, \quad k \in \{1, \dots, 15\}$$

or allowing for identity  $\eta = -\gamma^5$

$$(\Lambda'_k)^+ \gamma^5 = \gamma^5 \Lambda'_k, \quad k \in \{1, \dots, 15\}.$$

All generators  $\Lambda'_k$  of the group  $SU(2, 2)$  can be readily constructed on the basis of the known generators  $\Lambda_k$  of  $SU(4)$  (see (6.1))

$$\begin{aligned} \Lambda'_1 &= \Lambda_1 = \gamma^5, & \Lambda'_1 &= i\Lambda_1 = i\gamma^0, & \Lambda'_3 &= i\Lambda_3 = -\gamma^5\gamma^0, \\ \Lambda'_4 &= i\Lambda_4 = -\gamma^1, & \Lambda'_5 &= i\Lambda_5 = i\gamma^5\gamma^1, & \Lambda'_6 &= i\Lambda_6 = -\gamma^2, \\ \Lambda'_7 &= i\Lambda_7 = i\gamma^5\gamma^2, & \Lambda'_8 &= i\Lambda_8 = -\gamma^3, & \Lambda'_9 &= i\Lambda_9 = i\gamma^5\gamma^3, \\ \Lambda'_{10} &= \Lambda_{10} = 2\sigma^{01}, & \Lambda'_{11} &= \Lambda_{11} = 2\sigma^{02}, & \Lambda'_{12} &= \Lambda_{12} = 2\sigma^{03}, \\ \Lambda'_{13} &= \Lambda_{13} = 2i\sigma^{12}, & \Lambda'_{14} &= \Lambda_{14} = 2i\sigma^{23}, & \Lambda'_{15} &= \Lambda_{15} = 2i\sigma^{31}. \end{aligned} \quad (9.1)$$

Basis elements may be listed as follows:

$$\begin{aligned} \alpha'_1 &= \alpha_1 = \gamma^0\gamma^2, & \alpha'_2 &= i\alpha_2 = -\gamma^0\gamma^5, & \alpha'_3 &= i\alpha_3 = i\gamma^5\gamma^2 \\ (\alpha'_1)^2 &= I, & (\alpha'_2)^2 &= -I, & (\alpha'_3)^2 &= -I, \\ \alpha'_2\alpha'_3 &= -i\alpha_1, & \alpha'_3\alpha'_1 &= i\alpha'_2, & \alpha'_1\alpha'_2 &= i\alpha'_3; \\ \beta'_1 &= \beta_1 = i\gamma^3\gamma^1, & \beta'_2 &= i\beta_2 = -\gamma^3, & \beta'_3 &= i\beta_3 = -\gamma^1, \\ (\beta'_1)^2 &= I, & (\beta'_2)^2 &= -I, & (\beta'_3)^2 &= -I, \\ \beta'_2\beta'_3 &= -i\beta_1, & \beta'_3\beta'_1 &= i\beta'_2, & \beta'_1\beta'_2 &= i\beta'_3. \end{aligned}$$

These two sets commute with each other:  $\alpha_j\beta_k = \beta_k\alpha_j$ ; and their multiplications give remaining 9 elements:

$$\begin{aligned} A'_1 &= A_1 = \alpha'_1\beta'_1 = -\gamma^5, & B'_1 &= iB_1 = \alpha'_1\beta'_2 = i\gamma^5\gamma^1, & C'_1 &= iC_1 = \alpha'_1\beta'_3 = i\gamma^3\gamma^5, \\ A'_2 &= iA_2 = \alpha'_2\beta'_1 = \gamma^2, & B'_2 &= -B_2 = \alpha'_2\beta'_2 = +i\gamma^1\gamma^2, & C'_2 &= -C_2 = \alpha'_2\beta'_3 = +i\gamma^2\gamma^3, \\ A'_3 &= iA_3 = \alpha'_3\beta'_1 = i\gamma^0, & B'_3 &= -B_3 = \alpha'_3\beta'_2 = -\gamma^0\gamma^1, & C'_3 &= -C_3 = \alpha'_3\beta'_3 = -\gamma^0\gamma^3. \end{aligned}$$

Making in relations (7.6) a formal change in accordance with

$$\begin{aligned} e^{ia\Lambda} &= \cos a + i \sin a\Lambda, & \Lambda' &= i\Lambda, \\ \cos a + i \sin a\Lambda &= e^{ia\Lambda} \implies e^{ia\Lambda'} = \cos ia + i \sin ia\Lambda = \cosh a - \sinh a\Lambda, \end{aligned}$$

we arrive at explicit form of elementary pseudo-unitary  $SU(2, 2)$ -transformations:

$$\begin{aligned} \alpha'_1 &= \alpha_1 = \begin{vmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{vmatrix}, & U_1^\alpha &= \begin{vmatrix} \cos \phi + i \sin \phi \sigma_2 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_2 \end{vmatrix}, \\ \alpha'_2 &= i\alpha_2 = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, & U_2^\alpha &= \begin{vmatrix} \cosh \chi & -i \sinh \chi \\ i \sinh \chi & \cosh \chi \end{vmatrix}, \end{aligned}$$

$$\begin{aligned}
\alpha'_3 = i\alpha_3 &= \begin{vmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{vmatrix}, & U_3^\alpha &= \begin{vmatrix} \cosh \chi & -\sinh \chi \sigma_2 \\ \sinh \chi \sigma_2 & \cosh \chi \end{vmatrix}, \\
\beta'_1 = \beta_1 &= \begin{vmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{vmatrix}, & U_1^\beta &= \begin{vmatrix} \cos \phi + i \sin \phi \sigma_2 & 0 \\ 0 & \cos \phi + i \sin \phi \sigma_2 \end{vmatrix}, \\
\beta'_2 = i\beta_2 &= \begin{vmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{vmatrix}, & U_2^\beta &= \begin{vmatrix} \cosh \chi & i \sinh \chi \sigma_3 \\ -i \sinh \chi \sigma_3 & \cosh \chi \end{vmatrix}, \\
\beta'_3 = i\beta_3 &= \begin{vmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{vmatrix}, & U_3^\beta &= \begin{vmatrix} \cosh \chi & i \sinh \chi \sigma_1 \\ -i \sinh \chi \sigma_1 & \cosh \chi \end{vmatrix}, \\
A'_1 = A_1 &= \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, & U_1^A &= \begin{vmatrix} \cos \phi + i \sin \phi & 0 \\ 0 & \cos \phi - i \sin \phi \end{vmatrix}, \\
A'_2 = iA_2 &= \begin{vmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{vmatrix}, & U_2^A &= \begin{vmatrix} \cosh \chi & -i \sinh \chi \sigma_2 \\ i \sinh \chi \sigma_2 & \cosh \chi \end{vmatrix}, \\
A'_3 = iA_3 &= \begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix}, & U_3^A &= \begin{vmatrix} \cosh \chi & -\sinh \chi \\ -\sinh \chi & \cosh \chi \end{vmatrix}, \\
B'_1 = iB_1 &= \begin{vmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{vmatrix}, & U_1^B &= \begin{vmatrix} \cosh \chi & -\sinh \chi \sigma_1 \\ -\sinh \chi \sigma_1 & \cosh \chi \end{vmatrix}, \\
B'_2 = -B_2 &= \begin{vmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix}, & U_2^B &= \begin{vmatrix} \cos \phi + i \sin \phi \sigma_3 & 0 \\ 0 & \cos \phi + i \sin \phi \sigma_3 \end{vmatrix}, \\
B'_3 = -B_3 &= \begin{vmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{vmatrix}, & U_3^B &= \begin{vmatrix} \cos \phi + i \sin \phi \sigma_1 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_1 \end{vmatrix}, \\
C'_1 = iC_1 &= \begin{vmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{vmatrix}, & U_1^C &= \begin{vmatrix} \cosh \chi & \sinh \chi \sigma_3 \\ \sinh \chi \sigma_3 & \cosh \chi \end{vmatrix}, \\
C'_2 = -C_2 &= \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, & U_2^C &= \begin{vmatrix} \cos \phi + i \sin \phi \sigma_1 & 0 \\ 0 & \cos \phi + i \sin \phi \sigma_1 \end{vmatrix}, \\
C'_3 = -C_3 &= \begin{vmatrix} -\sigma_3 & 0 \\ 0 & +\sigma_3 \end{vmatrix}, & U_3^C &= \begin{vmatrix} \cos \phi - i \sin \phi \sigma_3 & 0 \\ 0 & \cos \phi + i \sin \phi \sigma_3 \end{vmatrix}. \tag{9.2}
\end{aligned}$$

We can easily obtain unitarity equations for  $SU(2, 2)$  group, simple solutions to which are given by (9.2). Indeed, taking into account the formulas (see (3.1))

$$\begin{aligned}
G^+ \eta &= \begin{vmatrix} k_0^* + \mathbf{k}^* \vec{\sigma} & -l_0^* - \mathbf{l}^* \vec{\sigma} \\ n_0^* - \mathbf{n}^* \vec{\sigma} & m_0^* - \mathbf{m}^* \vec{\sigma} \end{vmatrix} \begin{vmatrix} -I & 0 \\ 0 & +I \end{vmatrix} = \begin{vmatrix} -k_0^* - \mathbf{k}^* \vec{\sigma} & -l_0^* - \mathbf{l}^* \vec{\sigma} \\ -n_0^* + \mathbf{n}^* \vec{\sigma} & m_0^* - \mathbf{m}^* \vec{\sigma} \end{vmatrix}, \\
\eta G^{-1} &= \begin{vmatrix} -I & 0 \\ 0 & I \end{vmatrix} \begin{vmatrix} k'_0 + \mathbf{k}' \vec{\sigma} & n'_0 - \mathbf{n}' \vec{\sigma} \\ -l'_0 - \mathbf{l}' \vec{\sigma} & m'_0 - \mathbf{m}' \vec{\sigma} \end{vmatrix} = \begin{vmatrix} -k'_0 - \mathbf{k}' \vec{\sigma} & -n'_0 + \mathbf{n}' \vec{\sigma} \\ -l'_0 - \mathbf{l}' \vec{\sigma} & m'_0 - \mathbf{m}' \vec{\sigma} \end{vmatrix},
\end{aligned}$$

from  $G^+ \eta = \eta G^{-1}$  we produce

$$\begin{aligned}
k_0^* &= k'_0, & \mathbf{k}^* &= \mathbf{k}', & m_0^* &= m'_0, & \mathbf{m}^* &= \mathbf{m}', \\
l_0^* &= n'_0, & \mathbf{l}^* &= -\mathbf{n}', & n_0^* &= l'_0, & \mathbf{n}^* &= -\mathbf{l}'.
\end{aligned}$$

These relations differ from analogous ones (3.2) for  $SU(4)$  group only in all signs of the second line. Therefore, the unitarity conditions for the group  $SU(2, 2)$  are (compare with (3.3))

$$\begin{aligned}
k_0^* &= +k_0(mm) + m_0(ln) + l_0(nm) - n_0(lm) + i\mathbf{l}(\mathbf{m} \times \mathbf{n}), \\
m_0^* &= +m_0(kk) + k_0(nl) + n_0(lk) - l_0(nk) - i\mathbf{n}(\mathbf{k} \times \mathbf{l}), \\
\mathbf{k}^* &= -\mathbf{k}(mm) - \mathbf{m}(ln) - \mathbf{l}(nm) + \mathbf{n}(lm) + 2\mathbf{l} \times (\mathbf{n} \times \mathbf{m})
\end{aligned}$$

$$\begin{aligned}
& + im_0(\mathbf{n} \times \mathbf{l}) + il_0(\mathbf{n} \times \mathbf{m}) + in_0(\mathbf{l} \times \mathbf{m}), \\
\mathbf{m}^* &= -\mathbf{m}(kk) - \mathbf{k}(nl) - \mathbf{n}(lk) + \mathbf{l}(nk) + 2\mathbf{n} \times (\mathbf{l} \times \mathbf{k}) \\
& - ik_0(\mathbf{l} \times \mathbf{n}) - in_0(\mathbf{l} \times \mathbf{k}) - il_0(\mathbf{n} \times \mathbf{k}), \\
-l_0^* &= +k_0(nm) - m_0(kn) + l_0(nn) + n_0(km) + i\mathbf{k}(\mathbf{n} \times \mathbf{m}), \\
-n_0^* &= +m_0(lk) - k_0(ml) + n_0(ll) + l_0(mk) - i\mathbf{m}(\mathbf{l} \times \mathbf{k}), \\
-l^* &= -\mathbf{k}(nm) + \mathbf{m}(kn) - \mathbf{l}(nn) - \mathbf{n}(km) + 2\mathbf{k} \times (\mathbf{m} \times \mathbf{n}) \\
& + ik_0(\mathbf{m} \times \mathbf{n}) + im_0(\mathbf{k} \times \mathbf{n}) + in_0(\mathbf{m} \times \mathbf{k}), \\
-\mathbf{n}^* &= -\mathbf{m}(kl) + \mathbf{k}(ml) - \mathbf{n}(ll) - \mathbf{l}(mk) + 2\mathbf{m} \times (\mathbf{k} \times \mathbf{l}) - im_0(\mathbf{k} \times \mathbf{l}) \\
& - ik_0(\mathbf{m} \times \mathbf{l}) - il_0(\mathbf{k} \times \mathbf{m}).
\end{aligned}$$

Several words on factorization  $SU(2, 2) = SU(1, 1) \times [K \times L \times M] \times SU(1, 1)$  and a further group fine-structure for  $SU(2, 2)$ . On the basis of 9 matrices

$$\begin{aligned}
A'_1 = A_1 = \alpha'_1 \beta'_1 &= -\gamma^5, & B'_1 = iB_1 = \alpha'_1 \beta'_2 &= i\gamma^5 \gamma^1, & C'_1 = iC_1 = \alpha'_1 \beta'_3 &= i\gamma^3 \gamma^5, \\
A'_2 = iA_2 = \alpha'_2 \beta'_1 &= \gamma^2, & B'_2 = -B_2 = \alpha'_2 \beta'_2 &= +i\gamma^1 \gamma^2, & C'_2 = -C_2 = \alpha'_2 \beta'_3 &= i\gamma^2 \gamma^3, \\
A'_3 = iA_3 = \alpha'_3 \beta'_1 &= i\gamma^0, & B'_3 = -B_3 = \alpha'_3 \beta'_2 &= -\gamma^0 \gamma^1, & C'_3 = -C_3 = \alpha'_3 \beta'_3 &= -\gamma^0 \gamma^3
\end{aligned}$$

one can construct three 3-dimensional subsets (omitting three others):

$$\begin{aligned}
\mathbf{K}' &= \{A'_1 = A_1 = \alpha'_1 \beta'_1, B'_2 = -B_2 = \alpha'_2 \beta'_2, C'_3 = -C_3 = \alpha'_3 \beta'_3\}, \\
\mathbf{L}' &= \{C'_1 = iC_1 = \alpha'_1 \beta'_3, A'_2 = iA_2 = \alpha'_2 \beta'_1, B'_3 = -B_3 = \alpha'_3 \beta'_2\}, \\
\mathbf{M}' &= \{B'_1 = iB_1 = \alpha'_1 \beta'_2, C'_2 = -C_2 = \alpha'_2 \beta'_3, A'_3 = iA_3 = \alpha'_3 \beta'_1\},
\end{aligned}$$

with the same commutation relations:

$$\Gamma_1 \Gamma_2 = +\Gamma_3, \quad \Gamma_2 \Gamma_1 = +\Gamma_3, \quad \Gamma_1 \Gamma_2 - \Gamma_2 \Gamma_1 = 0, \quad \Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_1 = +2\Gamma_3,$$

and analogous ones by cyclic symmetry. Arbitrary element from  $SU(2, 2)$  can be factorized as follows

$$S = e^{i\vec{a}\vec{\alpha}'} e^{i\vec{b}\vec{\beta}'} e^{i\mathbf{k}\mathbf{K}'} e^{i\mathbf{l}\mathbf{L}'} e^{i\mathbf{m}\mathbf{M}'},$$

all parameters are real-valued. Let us specify the group law for these 5 sub-sets. Two groups  $e^{i\vec{a}\vec{\alpha}'}, e^{i\vec{b}\vec{\beta}'}$  are isomorphic so one can consider only the first one:

$$\begin{aligned}
e^{i\vec{a}\vec{\alpha}'} &= I + i(a_1\alpha'_1 + a_2\alpha'_2 + a_3\alpha'_3) - \frac{1}{2!}(a_1^2 - a_2^2 - a_3^2) \\
& - i\frac{1}{3!}(a_1^2 - a_2^2 - a_3^2)(a_1\alpha'_1 + a_2\alpha'_2 + a_3\alpha'_3) + \frac{1}{4!}(a_1^2 - a_2^2 - a_3^2)^2 + \dots
\end{aligned}$$

In the variables

$$a_i = an_i, \quad n_1^2 - n_2^2 - n_3^2 = 1$$

we have

$$\begin{aligned}
e^{i\vec{a}\vec{\alpha}'} &= I + i(n_1\alpha'_1 + n_2\alpha'_2 + n_3\alpha'_3)a - \frac{1}{2!}a^2 - i\frac{1}{3!}(n_1\alpha'_1 + n_2\alpha'_2 + n_3\alpha'_3)a^3 \\
& + \frac{1}{4!}a^4 + \frac{1}{5!}(n_1\alpha'_1 + n_2\alpha'_2 + in_3\alpha'_3)a^5 - \frac{1}{6!}a^6 + \dots,
\end{aligned}$$

that is

$$e^{i\vec{a}\vec{\alpha}'} = \cos a + i \sin a (n_1\alpha'_1 + n_2\alpha'_2 + n_3\alpha'_3) = x_0 - ix_1\alpha'_1 - ix_2\alpha'_2 - ix_3\alpha'_3.$$

Multiplying two matrices we arrive at

$$\begin{aligned} x''_0 &= x'_0 x_0 - x'_1 x_1 + x'_2 x_2 + x'_3 x_3, & x''_1 &= x'_0 x_1 + x'_1 x_0 - (x'_2 x_3 - x'_3 x_2), \\ x''_2 &= x'_0 x_2 + x'_2 x_0 + (x'_3 x_1 - x'_1 x_3), & x''_3 &= x'_0 x_3 + x'_3 x_0 + (x'_1 x_2 - x'_2 x_1). \end{aligned}$$

The inverse matrix looks

$$(x_0, \mathbf{x})^{-1} = (x_0, -\mathbf{x}).$$

Parameters  $(x_0, x_i)$  should obey the following condition

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 1.$$

In the variables  $a, n_i$  it will look

$$\cos^2 a + \sin^2 a (n_1^2 - n_2^2 - n_3^2) = 1, \quad n_1^2 - n_2^2 - n_3^2 = 1.$$

For three particular cases we will have:

$$\begin{aligned} \mathbf{n} &= (1, 0, 0), & \cos^2 a + \sin^2 a &= 1, & e^{ia\alpha'_1} &= \cos a + i \sin a \alpha'_1, & a &\in R; \\ \mathbf{n} &= (0, i, 0), & \cos^2 a - \sin^2 a &= 1, & & & & \\ a &= ib, & \cos ib &= \cosh b, & \sin ib &= i \sinh b, & b &\in R, \\ e^{ia\alpha'_2} &= \cos a + i \sin a n_2 \alpha'_2 = \cosh b - i \sinh b \alpha'_2; \\ \mathbf{n} &= (0, 0, i), & \cos^2 a - \sin^2 a &= 1, & & & & \\ a &= ib, & \cos ib &= \cosh b, & \sin ib &= i \sinh b, & b &\in R, \\ e^{ia\alpha'_3} &= \cos a + i \sin a n_3 \alpha'_3 = \cosh b - i \sinh b \alpha'_3. \end{aligned}$$

Now let us turn to finite transformations from remaining sub-sets  $\mathbf{K}'$ ,  $\mathbf{L}'$ ,  $\mathbf{M}'$ . Each of tree subgroups can be constructed as multiplying of elementary 1-parametric commuting transformations. Their explicit forms are:

subgroup  $K'$

$$\begin{aligned} \mathbf{K}' &= \{A'_1 = A_1 = \alpha'_1 \beta'_1, B'_2 = -B_2 = \alpha'_2 \beta'_2, C'_3 = -C_3 = \alpha'_3 \beta'_3\}, \\ A'_1 &= \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \quad B'_2 = \begin{vmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix}, \quad C'_3 = \begin{vmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix}; \end{aligned}$$

subgroup  $L'$

$$\begin{aligned} \mathbf{L}' &= \{C'_1 = iC_1 = \alpha'_1 \beta'_3, A'_2 = iA_2 = \alpha'_2 \beta'_1, B'_3 = -B_3 = \alpha'_3 \beta'_2\}, \\ C'_1 &= \begin{vmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{vmatrix}, \quad A'_2 = \begin{vmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{vmatrix}, \quad B'_3 = \begin{vmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{vmatrix}; \end{aligned}$$

subgroup  $M'$

$$\begin{aligned} \mathbf{M}' &= \{B'_1 = iB_1 = \alpha_1 \beta_2, C'_2 = -C_2 = \alpha_2 \beta_3, A'_3 = iA_3 = \alpha_3 \beta_1\}, \\ B'_1 &= \begin{vmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{vmatrix}, \quad C'_2 = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, \quad A'_3 = \begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix}. \end{aligned}$$

On the basis of 15 matrices

$$\alpha'_1, \quad \alpha'_2, \quad \alpha'_3, \quad \beta'_1, \quad \beta'_2, \quad \beta'_3,$$

$$\begin{aligned}
A'_1 &= \alpha'_1 \beta'_1, & B'_1 &= \alpha'_1 \beta'_2, & C'_1 &= \alpha'_1 \beta'_3, \\
A'_2 &= \alpha'_2 \beta'_1, & B'_2 &= \alpha'_2 \beta'_2, & C'_2 &= \alpha'_2 \beta'_3, \\
A'_3 &= \alpha'_3 \beta'_1, & B'_3 &= \alpha'_3 \beta'_2, & C'_3 &= \alpha'_3 \beta'_3
\end{aligned}$$

one can easily see the following 20 ways to separate  $SU(1, 1)$  subgroups:

$$\begin{aligned}
&(\alpha'_1, \alpha'_2, \alpha'_3), & (\beta'_1, \beta'_2, \beta'_3), \\
&(\alpha'_1, A'_2, A'_3), & (A'_1, \alpha'_2, A'_3), & (A'_1, A'_2, \alpha'_3), \\
&(\alpha'_1, B'_2, B'_3), & (B'_1, \alpha'_2, B'_3), & (B'_1, B'_2, \alpha'_3), \\
&(\alpha'_1, C'_2, C'_3), & (C'_1, \alpha'_2, C'_3), & (C'_1, C'_2, \alpha'_3), \\
&(\beta'_1, B'_1, C'_1), & (\beta'_1, B'_2, C'_2), & (\beta'_1, B'_3, C'_3), \\
&(A'_1, \beta'_2, C'_1), & (A'_2, \beta'_2, C'_2), & (A'_3, \beta'_2, C'_3), \\
&(A_1, B_1, \beta_3), & (A_2, B_2, \beta_3), & (A_3, B_3, \beta_3).
\end{aligned}$$

such 3-subgroups might be used as bigger elementary blocks in constructing a general transformation.

## 10 On pseudo-unitary group $SU(3, 1)$

Let us show how the above formalism can apply to the pseudo-unitary group  $SU(3, 1)$ . Transformations from  $SU(3, 1)$  should leave invariant the following form

$$(z_1^*, z_2^*, z_3^*, z_4^*) \begin{vmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{vmatrix}, \quad z^+ \eta z = z'^+ \eta z',$$

which leads to

$$z' = U z, \quad \text{where} \quad U^+ \eta = \eta U^{-1}, \quad \eta = \begin{vmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}.$$

Any generator  $\Lambda'$  of those transformations must obey the relation

$$U_k = e^{i a \Lambda'_k}, \quad (\Lambda'_k)^+ \eta = \eta \Lambda'_k. \quad k \in \{1, \dots, 15\}.$$

The matrix  $\eta$  is a linear combination

$$\eta = \frac{1}{2}(2i\sigma^{12} - 2\sigma^{03} - \gamma^5 + I) = \frac{1}{2}(i\gamma^1\gamma^2 - \gamma^0\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3 + I).$$

All generators of the group  $SU(3, 1)$  can readily be constructed on the basis of the known generators  $\lambda_k$  of  $SU(4)$  (see (6.2))

$$\begin{aligned}
&\lambda_1, & \lambda_2, & \lambda_3, & \lambda_4, & \lambda_5, & \lambda_6, & \lambda_7, & \lambda_8, & i\lambda_9, & i\lambda_{10}, \\
&i\lambda_{11}, & i\lambda_{12}, & i\lambda_{13}, & i\lambda_{14}, & \lambda_{15};
\end{aligned} \tag{10.1}$$

generator  $\lambda_9, \dots, \lambda_{14}$  are multiplied by imaginary unit  $i$ . Instead of  $\lambda_8, \lambda_{15}$  one can introduce other generator  $\lambda'_8, \lambda'_{15}$  see (6.4) (diagonal generators are the same for group  $SU(4)$ ,  $SU(2, 2)$ , and  $SU(3, 1)$ ).

## 11 Discussion

Let us summarize the main point of the present treatment.

Parametrization of  $4 \times 4$  matrices  $G$  of the complex linear group  $GL(4, C)$  in terms of four complex 4-vector parameters  $(k, m, n, l)$  is investigated. Additional restrictions separating some subgroups of  $GL(4, C)$  are given explicitly. In the given parametrization, the problem of inverting any  $4 \times 4$  matrix  $G$  is solved. Expression for determinant of any matrix  $G$  is found:  $\det G = F(k, m, n, l)$ . Unitarity conditions  $G^+ = G^{-1}$  have been formulated in the form of non-linear cubic algebraic equations including complex conjugation. Several simplest solutions of these unitarity equations have been found: three 2-parametric subgroups  $G_1, G_2, G_3$  – each of subgroups consists of two commuting Abelian unitary groups; 4-parametric unitary subgroup consisting of a product of a 3-parametric group isomorphic  $SU(2)$  and 1-parametric Abelian group.

The Dirac basis of generators  $\Lambda_k$ , being of Gell-Mann type, substantially differs from the basis  $\lambda_i$  used in the literature on  $SU(4)$  group, formulas relating them are found – they permit to separate  $SU(3)$  subgroup in  $SU(4)$ . Special way to list 15 Dirac generators of  $GL(4, C)$  can be used  $\{\Lambda_k\} = \{\alpha_i \oplus \beta_j \oplus (\alpha_i \beta_j = \mathbf{K} \oplus \mathbf{L} \oplus \mathbf{M})\}$ , which permit to factorize  $SU(4)$  transformations according to  $S = e^{i\vec{a}\vec{\alpha}} e^{i\vec{b}\vec{\beta}} e^{i\vec{k}\vec{K}} e^{i\vec{l}\vec{L}} e^{i\vec{m}\vec{M}}$ , where two first factors commute with each other and are isomorphic to  $SU(2)$  group, the three last are 3-parametric groups, each of them consists of three Abelian commuting unitary subgroups. Besides, the structure of fifteen Dirac matrices  $\Lambda_k$  permits to separate twenty 3-parametric subgroups in  $SU(4)$  isomorphic to  $SU(2)$ ; those subgroups might be used as bigger elementary blocks in constructing a general transformation  $SU(4)$ . It is shown how one can specify the present approach for the unitary group  $SU(2, 2)$  and  $SU(3, 1)$ .

In principle, all different approaches used in the literature are closely related so that any result obtained within one technique may be easily translated to any other. There is no sense to persist in exploiting only one representation, thinking that it is much better than all others. Success should lie in combining different techniques. For instance, Euler angles-based approach provides us with the group elements in the separated variables form, which may be of a supreme importance at calculating matrix elements of the group. In turn, a factorized subgroup-based structure is of special interest in the particle physics and gauge theory of fundamental interaction. Geometrical properties of the groups, their global structure, differences between orthogonal groups and their double covering, and so on, seem to be most easily understood in terms of bilinear functions in space of linear parameters:  $G = x_j \Lambda_j$ ,  $x''j = e_{jkl} x'_k x_l$ .

We have no ground to think that only exponential functions  $e^{i\Lambda}$  are suitable for exploration into group structures. We may expect that in addition to Euler angles many other curvilinear coordinates might be of value for studying of the group structure. For instance, in the case of the group  $SO(4, C)$  we have known 34 such coordinate systems owing to Olevskiy investigation [58] on 3-orthogonal coordinates in real Lobachevski space.

In conclusion, several words about possible application areas of the obtained results. The main argument in favor of constructing the theory of unitary groups  $SU(4)$  (and related to it) in terms of Dirac matrices is the role of spinor methods being widely adopted in physics. Let us mention several problems most attractive for authors:

- $SU(2, 2)$  and conformal symmetry, massless particles;
- classical Yang-Mills equations and gauge fields;
- geometric phases for multi-level quantum systems;
- composite structure of quarks and leptons;
- $SU(4)$  gauge models.

In particular, description of the group  $SU(2, 2)$  in terms of matrices  $\alpha^j$ ,  $\beta^j$  should be of great benefit in investigation of *conformal symmetry in massless particles theory*. For instance, classical Maxwell equations in a medium can be presented in 4-dimensional complex matrix form with the use of two sets of matrices, exploited above:

$$\left(-i\frac{\partial}{\partial x^0} + \alpha^j \frac{\partial}{\partial x^j}\right) M + \left(-i\frac{\partial}{\partial x^0} + \beta^j \frac{\partial}{\partial x^j}\right) N = \frac{1}{\epsilon_0} \left| \begin{array}{c} \rho \\ \mathbf{j}/c \end{array} \right|,$$

where

$$M = \left| \begin{array}{c} 0 \\ \mathbf{M} \end{array} \right|, \quad \mathbf{M} = \frac{1}{\epsilon_0}(\mathbf{D} + i\mathbf{H}/c) + (\mathbf{E} + ic\mathbf{B}),$$

$$N = \left| \begin{array}{c} 0 \\ \mathbf{N} \end{array} \right|, \quad \mathbf{N} = \frac{1}{\epsilon_0}(\mathbf{D} - i\mathbf{H}/c) - (\mathbf{E} - ic\mathbf{B}).$$

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